Aspects of gauge/gravity duality and it's application

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Dedicated to my parents...

Balai Chandra Ghorai and Sikha Rani Ghorai

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List of Publications

The thesis is based on the followings papers

1. **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Higher dimensional holographic superconductors in Born–Infeld electrodynamics with back-reaction"*, **Eur. Phys. J. C 76 (2016) 146.**

2. **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Holographic free energy and thermodynamic geometry"*, **Eur. Phys. J. C 76 (2016) 702.**

3. **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Non-linear effects on the holographic free energy and thermodynamic geometry"*, **Euro Phys. Lett. 118 (2017) 31001.**

4. **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Conductivity of holographic superconductors in Born–Infeld electrodynamics"*, **Nucl. Phys. B 933 (2018) 1-13.**

5. **Debabrata Ghorai**, Sunandan Gangopadhyay, Rabin Banerjee, *"Meissner effect in holographic superconductors with Dirac-Born-Infeld electrodynamics"*, **Mod. Phys. Lett. A 35 (2019) 2050020.**

Publications not included in the thesis

6. **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Noncommutative effects of spacetime on holographic superconductors"*, **Phys. Lett. B 758 (2016) 106.**

7. Saurav Das, Sunandan Gangopadhyay, **Debabrata Ghorai**, *"Viscosity to entropy density ratio for non-extremal Gauss-Bonnet black holes coupled to Born-Infeld electrodynamics"*, **Eur. Phys. J. C 77 (2017) 615.**

8. Diganta Parai, **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Noncommutative effects of charged black hole on holographic superconductors"*, **Gen. Rel. Grav. 50 (2018) 149.**

9. Ravikant Verma, **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Path integral action of a particle in κ-Minkowski spacetime"*, **Euro Phys. Lett. 122 (2018) 40001.**

10. Sourav Karar, **Debabrata Ghorai**, Sunandan Gangopadhyay, *"Holographic entanglement thermodynamics for higher dimensional charged black hole"*, **Nucl. Phys. B 938 (2019) 363.**

11. Ankur Srivastav, **Debabrata Ghorai**, Sunandan Gangopadhyay, *"p-wave holographic superconductors with massive vector condensate in Born-Infeld electrodynamics"*, **Eur. Phys. J. C 80 (2020) 219.**

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Chapter 1

Introduction and Overview

1.1 Introduction

Nature is governed by four fundamental forces, namely, gravity, electromagnetic force, weak force and strong force. Any physical phenomena in Nature is described by any of these interactions. The strong bonding in nucleus of an atom is described by strong interaction and decay from nucleus of atom is explained by weak interaction. Most of the physical processes and the attraction between charges is described by electromagnetic interaction. The underlying root of these three interaction lie on the gauge theory which is based on the quantum version of physical phenomena. The attraction between two objects (comparable large) is describe by gravity which is contrasted with quantum mechanics. The unification of all four fundamental interactions is the biggest challenge since 19th-century. This unification is important because all physical processes in Nature could be explained in terms of one theory. The quantum version of gravity theory will also help to understand the property of spacetime (continuous or discrete), origin of universe and black hole physics in deeper way. The main problem with gravity theory is that it is not renormalizable theory. Linearized gravity with quantum mechanics leads to graviton which is massless spin-2 particle. This hints us that some kind of unification between gravity and gauge theory is possible if massless spin-2 particle exist in any gauge theory or we construct a quantum field theory (QFT) with massless spin-2 particle. In quantum chromodynamics (QCD), massive spin-2 particle exist, which suggest to construct QCD like theory for massless spin-2 particle. No-go theorem [1] forbids this possibility, which tells us that a theory with Lorentz covariant conserved energy momenta tensor $T^{\mu\nu}$ can not contain massless particle of spin greater than 1. This suggests that massive spin-2 particles can never become massless spin-2 particles in same spacetime dimension. This hints that one can incorporate massless spin-2 particle in a QFT if this particle lives in other spacetime dimension. This leads to a hint of connection with holographic principle [2],[3] in the context of black hole physics. It is then believed that quantum gravity can be realized by studying dual field theory in lower dimensional curved spacetime. The dual theory (Maxwell's duality, Bosonization, S,T-duality) works very well to study physical system. All these hints establish that there may be a possible duality between gauge(QFT) and gravity theory. Maldacena [4] showed from framework of string theory that $\mathcal{N} = 4$ super

Yang-Mills theory in four dimensional spacetime is dual to type IIB string theory on $AdS_5 \times S^5$ at low energies. This shows us that there is exact mapping between a gravity theory in AdS_5 and conformal field theory (CFT) on boundary of that spacetime, which is known as AdS/CFT correspondence. This conjecture was then realized as gauge/gravity duality [5]-[7] which is the generalization of AdS/CFT correspondence. The holographic principle, gauge/gravity duality and AdS/CFT correspondence are related by set relation in following way [8]

$$AdS/CFT\ correspondence\ \subset\ gauge/gravity\ duality\ \subset\ holographic\ principle.$$

The gauge/gravity duality [4]-[7] provides an exact correspondence between gravity theory in a $(d+1)$-dimensional AdS spacetime and a gauge theory sitting on the d-dimensional boundary of this spacetime. It has remarkable ability to address issues in strongly interacting systems by exploiting results obtained in a weakly coupled gravitational system. This duality has been extracted from string theory which provides a framework to tackle strongly coupled system using weakly coupled variables. At low-energies, type IIB string theory on $AdS_5 \times S^5$ reduces to type IIB supergravity on $AdS_5 \times S^5$. The equivalence between type IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM) $SU(N)$ gauge theory in the large N limit has been very well tested. This idea is related to large N gauge theory which is equivalent to a string theory.

This is holographic because it states that gravity in five dimensions is equivalent to a local field theory in four dimensions. Using gravitational dual of strongly coupled system one may explain some of its properties which may in turn give some insight in the intricacies of the duality itself. In the past two decades, the dual gravitational theories played a very important role in theoretical physics to study quantum chromodynamics [9]-[13], fluid dynamics [14]-[21], entanglement entropy [22]-[29] and condensed matter physics [30]-[32]. Using spontaneous symmetry breaking [33] in gravity theory and this duality, a holographic description of properties of high T_c superconductor has been given in [34] which is known as holographic superconductor. The name suggests that the $d+1$-dimensional Einstein gravity theory resembles some basic properties of d-dimensional superconductors, which involves a charged black hole with non-trivial "scalar hair" (a static non-zero field outside a black hole). Asymptotic Anti-de Sitter spacetime plays crucial role for the formation of scalar hair. Using instability of scalar hair [33], the ratio between the band gap energy and critical temperature of superconductors $\frac{\Delta_0}{k_B T_c} = 4.2$ has been computed numerically in [34] with help of AdS/CFT dictionary, which matches with high T_c superconductors (~ 3.78). This gravitational dual reproduces some basic properties of superconductors. Another interesting development in this field has been the finding of the universal value $\frac{1}{4\pi}$ for the ratio of the shear viscosity η to the entropy density s [14]. The dual gravity descriptions in which the computation was carried out a variety of theories described by Einstein gravity and GB gravity coupled to Maxwell electrodynamics. The gauge/gravity correspondence has also played a key role in computing the entanglement entropy of a boundary conformal field theory holographically from its bulk gravitational dual, which is known as holographic entanglement entropy [22]. The insight comes from the fact that the holographic principle states that the number of degrees of freedom in a region of space is equal in number to the degrees of freedom on the boundary that surrounds the space. The prescription of computing the holographic entanglement entropy (HEE) was first proposed in [22] which involves with the minimal surface area of the bulk extension whose boundary coincides with the edges of the subsystem living at the boundary.

This thesis is based on the application of gauge/gravity duality in different systems which are strongly coupled. The perturbation theory is a powerful tool to analyze physical systems which are weakly coupled. This powerful tool works remarkably on all systems except strongly coupled system. There is no unique way to analyze strongly coupled systems because all conventional tools fail to describe these systems. The gauge/gravity duality is more general version of AdS/CFT correspondence which is a powerful tool to describe strongly coupled system by analyzing it's dual weakly coupled system. This duality maps between quantities in weakly coupled gravity theory and quantities in strongly coupled gauge theory. Exploiting this duality, strongly coupled systems have been investigated in this thesis. We have mainly focused on the analytical investigation of higher dimensional holographic superconductors in presence of Born-Infeld electrodynamics. To describe some properties of this strongly coupled gauge theory, we have considered different gravity theory (Einstein gravity and Gauss-Bonnet (GB) gravity) with different electrodynamics (Maxwell electrodynamics and Born-Infeld (BI) electrodynamics). The critical properties, conductivity of holographic superconductors and response of external magnetic field have been investigated with different analytical methods [35]-[39]. The importance of higher dimensional gravity theory (Gauss-Bonnet gravity) and non-linear electrodynamics (Born-Infeld electrodynamics) is to know the different non-linear effects on the system. There are other important properties which leads to consider these non-linear theories in the context of application of gauge/gravity duality, will be discussed in subsequent chapter. Before discussing the overview of thesis, we would like to mention brief discussion on general relativity, black hole thermodynamics and Born-Infeld electrodynamics in next subsequent sections. This brief introduction makes stronger the pillar of gauge/gravity duality and its applications.

1.2 Brief discussion on general relativity

General Relativity (GR) was established by Albert Einstein in 1916 [40]-[42], which is a remarkably successful theory of gravity to understand mysteries of our universe. The predictions of this theory is well established by observations and Newton's theory of gravitation is retrieved in non-relativistic limit from GR. After hundred years of discovery of GR, two major predictions, gravitational waves and black hole are directly verified by observation in LIGO [43] and EHT [44],[45] respectively. Before discovery of GR, Einstein established theory special relativity [46] in which he treated space and time as single quantity spacetime and defined the line element of spacetime as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

where we have used notation $x^\mu \equiv (t, x, y, x) = (x^0, x^1, x^2, x^3)$ and considered $\eta_{\mu\nu} = (-, +, +, +)$ convention with natural unit ($c = 1, k_B = 1, \hbar = 1$). Special relativity is based on the principle of relativity which excludes accelerated motion. GR is based on the generalization of principle of relativity, the principle of general covariance, the equivalence principle, the principle of causality. GR tells that the effects of gravity are mimicked by geometry. The equivalence principle establishes the exact relation between geometry and matter. In the presence of matter, the geometry of the spacetime is curved and it effects the line element of the spacetime in following way [46]

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{1.2}$$

where $g_{\mu\nu}(x)$ is determined in the presence of matter. To establish the above claim, we start with Newtonian equation of motion. From equivalence principle, the Newtonian equation can simplified as

$$m_I \frac{d^2\vec{x}}{dt^2} = -m_G \vec{\nabla}\phi \quad \Rightarrow \quad \frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi \tag{1.3}$$

where $\phi = -\frac{GM}{|\vec{x}|}$ is gravitational potential. The Poisson's equation for gravity is

$$\nabla^2\phi = 4\pi G\rho \tag{1.4}$$

where ρ is the matter density and G is the Newtonian constant. This equation tells us about instantaneous event. We know from special theory of relativity that nothing moves faster than light which tells that this equation is problematic. One can modify this equation by wave operator $\Box$. But this does not explain several phenomena like bending of light. To know the effect of matter on geometry, we consider a thought experiment [47]. Lets consider two clock siting at different places in a static gravitational field. One clock sits at height L from ground whereas another clock sits at ground in which line element is flat. This two clocks measure different time for any event because of the gravitational field. Consider a photon gun emitting a photon which is received at height L. From conservation of energy we can write

$$E_{ground} = E_h + mgL \tag{1.5}$$

where m is the mass of particle. For single photon, the above equation becomes

$$h\nu_{down} = h\nu_{up} + \frac{h\nu_{up}}{c^2} gL \quad \Rightarrow \quad \nu_{down} = \nu_{up}\left(1 + \frac{gL}{c^2}\right) \tag{1.6}$$

which leads to

$$(\Delta t)_{up} = (\Delta t)_{down}\left(1 + \frac{gL}{c^2}\right) . \tag{1.7}$$

In the second term, gL can be identified as the potential difference between ground and height L which can be written as $gL = \phi_{up} - \phi_{down}$. In the above calculations we assume gravitational potential energy is zero at ground. So gL actually represents the gravitational potential ϕ. For any arbitrary distance (height x), this potential is represented by $\phi(x)$. Substituted in above equation, we obtain

$$(\Delta t)_x = (\Delta t)_{ground}\left(1+\frac{\phi(x)}{c^2}\right) \quad \Rightarrow (\Delta t)_x^2 \approx (\Delta t)_{ground}^2\left(1+\frac{2\phi(x)}{c^2}\right) \tag{1.8}$$

Due to gravity, the clock at height x measures a different time which leads to modify the line element as (put $c = 1$ for natural unit)

$$ds^2 = -\left(1+2\phi(x)\right)(dt)^2 + (d\vec{x})^2 \ . \tag{1.9}$$

So we start with flat metric at ground where gravitational potential is zero. When photon moves through earth's gravitational potential, the metric takes into form as

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{1.10}$$

where $g_{00} = -\left(1+2\phi(x)\right) = -\left(1-\frac{2GM}{x}\right)$.
We now want to construct the Einstein-Hilbert action for gravity. We start with the discussion of a free particle motion in absence of gravity and in presence of gravity. The action for free particle in Minkowski spacetime (flat spacetime) is given by

$$S = \int ds = \int d\tau\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \ . \tag{1.11}$$

Extremizing the above equation we find the equation of motion for free particle in flat spactime which reads

$$\frac{d^2x^\mu}{d\tau^2} = 0 \ . \tag{1.12}$$

This is invariant under any linear transformations which is generally known as inhomogeneous Lorentz transformation ($x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$). We now want to see how the equation of motion of free particle changes under general coordinate transformation

$$x^\mu \to x'^\mu = x'^\mu(x^\nu).$$

Under this transformation the eq.(1.12) transforms as

$$\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\nu}{\partial x'^\mu}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \ . \tag{1.13}$$

For linear transformation the extra piece vanishes. In general coordinate transformation, the equation of motion for free particle is modified by this extra term. This extra term is identified as the Christoffel connection term in presence of gravity. To identify this extra term, we have generalized the action for free particle in presence of gravity by putting $g_{\mu\nu}$ in eq.(1.11)

$$S_g = \int d\tau\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \ . \tag{1.14}$$

Extremizing this, we find the equation of motion for free particle in presence of gravity

$$\frac{d^2x^\nu}{d\tau^2} + {\Gamma^\nu}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{1.15}$$

$${\Gamma^\nu}_{\rho\sigma} = \frac{1}{2}g^{\nu\alpha}(-\partial_\alpha g_{\rho\sigma} + \partial_\sigma g_{\alpha\rho} + \partial_\rho g_{\alpha\sigma}) \tag{1.16}$$

where ${\Gamma^\nu}_{\rho\sigma}$ is known as Christoffel connection and the eq.(1.15) is called Geodesic equation. Because of the principle of general covariance, we need to define covariant derivative in which the covariant derivative of a tensor transforms as a tensor. A tensor is defined by following transformation under general coordinate transformation

$${A^{\mu_1....\mu_n}}_{\nu_1....\nu_n} \rightarrow {A'^{\mu_1....\mu_n}}_{\nu_1....\nu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}....\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}....\frac{\partial x^{\sigma_n}}{\partial x'^{\nu_n}}{A^{\rho_1....\rho_n}}_{\sigma_1....\sigma_n} .$$

The covariant derivatives of the vector and 1-form are given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + V^\alpha {\Gamma^\nu}_{\mu\alpha} \qquad \& \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu - V_\alpha {\Gamma^\alpha}_{\mu\nu} . \tag{1.17}$$

Preserving the covariance structure, one can construct Riemann tensor

$${R^\mu}_{\nu\rho\sigma} = \partial_\sigma \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma}$$

from a metric. Ricci tensor ($R_{\mu\nu}$) and Ricci scalar (R) are constructed from the above expression. They are defined in following way

$$R_{\mu\nu} = {R^\rho}_{\mu\rho\nu} \qquad \& \qquad R = {R^\mu}_\mu = g^{\mu\nu}R_{\mu\nu} .$$

Using this all possible tensor, we shall try to write down the Einstein field equation by argument from [48]. From this Einstein field equation, we can construct the action of gravity which must be a scalar under general coordinate transformation.

We now try to generalize (relativistic) the Poisson equation for Newtonian potential (1.4). The left hand side (LHS) of the eq.(1.4) is the second order derivative of potential which is the measure of the effect of mass distribution ρ. The right hand side(RHS) of that equation is the measure of mass distribution which can be generalized in relativistic form as energy-momentum tensor ($T_{\mu\nu}$). The relativistic generalization of LHS of that equation should be proportional to second rank tensor which can be constructed from second derivative of metric since gravity is mimicked by geometry. In this way we identify that LHS side of Einstein field equation should be involved only in geometry and RHS of Einstein field equation should be involved in matter part. The first guess should be metric $g_{\mu\nu}$ itself in following way

$$\left[\nabla^2 g\right]_{\mu\nu} \propto T_{\mu\nu}$$

which is not sensible due to metric compatibility ($\nabla^2 g_{\mu\nu} = 0$). The second guess : a second rank tensor which is proportion to $T_{\mu\nu}$, could be Ricci tensor ($R_{\mu\nu}$)

$$R_{\mu\nu} \propto T_{\mu\nu} . \tag{1.18}$$

This guess also does not work because of the conservation of energy-momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$). From Bianchi identity we know that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \tag{1.19}$$

which implies that if we take covariant derivative of both side of above equation, they are inconsistent. Using above two eq.(s)(1.18,1.19), we can construct a second rank tensor in following way

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{1.20}$$

which is known as Einstein field tensor. We now get the Einstein field equation as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \propto T_{\mu\nu} \tag{1.21}$$

which is consistent with conservation of energy-momentum tensor. The proportionality constant is identified by taking non-relativistic limit and using Newtonian gravity. With identification of proportionality constant, the Einstein field equation now reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.22}$$

where $T_{\mu\nu}$ is the energy-momentum tensor in presence of matter. For vacuum, $T_{\mu\nu} = 0$ which leads to the Schwarzschild black hole solution.

We now can construct the action from Einstein field equation by back calculation which turns out as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R \tag{1.23}$$

where $\kappa^2 = 8\pi G$ and $\sqrt{-g}d^4x$ is the volume element which is invariant under general coordinate transformation. This is known as the Einstein-Hilbert action which is scalar quantity as required for any action. The variation of the above action leads to

$$\begin{aligned} \delta S &= \frac{1}{2\kappa^2}\int d^4x\left[\sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu}) + (\delta\sqrt{-g})R_{\mu\nu}g^{\mu\nu} + \sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu}\right] \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] + surface\ term \end{aligned} \tag{1.24}$$

This gives us to Einstein equation without matter which reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\ . \tag{1.25}$$

When we add matter action S_{matter}, the above equation becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.26}$$

where $T_{\mu\nu}$ is the energy-momentum tensor takes form as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \ . \tag{1.27}$$

The Einstein-Hilbert action becomes in presence of cosmological constant Λ [48]

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(R - 2\Lambda) \tag{1.28}$$

which leads to Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \ . \tag{1.29}$$

This gravity theory was generalized in higher dimension spacetime with higher curvature correction [49] in following way

$$S_{GB} = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}\left(R + \frac{\alpha}{2}\left[R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\right]\right) \tag{1.30}$$

which is known as Gauss-Bonnet (GB) gravity in d-dimensional spacetime and α is called GB parameter. The field equation from this action yields

$$\begin{aligned} R_{\mu\nu} \ &- \ \frac{1}{2}g_{\mu\nu}R - \frac{\alpha}{4}g_{\mu\nu}\left[R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\right] \\ &- \ \frac{\alpha}{2}\left[-2RR_{\mu\nu} + 4R^{\alpha\beta}R_{\alpha\mu\beta\nu} - 2R_{\mu\alpha\beta\gamma}R_{\nu}^{\ \alpha\beta\gamma} + 4R_{\mu}^{\ \alpha}R_{\mu\alpha}\right] = 0 \end{aligned} \tag{1.31}$$

This generalized gravity theory plays crucial role in phase transition of gravity theory. The importance of this theory in context of holographic superconductors is shown in chapter 3.

1.2.1 Black hole from Einstein field equation

The assumption of the suitable symmetry plays crucial role to achieve the exact solution of Einstein field equation. For instance, the static and the spherical symmetry of the spacetime leads to the well-known Schwarzschild solution which is the solution of Einstein field equation in the absence of matter distribution. With our signature convention $(-,+,+,+)$, the metric with the static and spherical symmetry takes form in spherical coordinate

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ . \tag{1.32}$$

Since the metric is static, the unknown coefficient g_{tt} and g_{rr} are function of r only. With this, the metric (1.32) becomes

$$ds^2 = -f(r)dr^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.33}$$

where $f(r)$ and $h(r)$ are to be determined by the Einstein field equation. With this metric, we can calculate the Christoffel symbols which leads us the Ricci tensor $R_{\mu\nu}$

and Ricci scalar R as

$$
\begin{aligned}
R_{tt} &= \frac{-\frac{f'(r)^2}{4f(r)}+\frac{f'(r)}{r}+\frac{f''(r)}{2}}{h(r)} - \frac{f'(r)h'(r)}{4h(r)^2} \\
R_{rr} &= \frac{f(r)(4f(r)+rf'(r))h'(r)+rh(r)\left(f'(r)^2-2f(r)f''(r)\right)}{4rf(r)^2h(r)} \\
R_{\theta\theta} &= \frac{1}{2}\left(-\frac{\frac{rf'(r)}{f(r)}+2}{h(r)}+\frac{rh'(r)}{h(r)^2}+2\right) \\
R_{\phi\phi} &= \frac{\sin^2(\theta)\left(f(r)\left(2h(r)^2-2h(r)+rh'(r)\right)-rh(r)f'(r)\right)}{2f(r)h(r)^2}
\end{aligned}
\tag{1.34}
$$

$$
\begin{aligned}
R = \frac{1}{2r^2f(r)^2h(r)^2} &\times \left\{r^2h(r)f'(r)^2 + rf(r)\left(rf'(r)h'(r) - 2h(r)\left(rf''(r)+2f'(r)\right)\right)\right. \\
&+ \left. 4f(r)^2\left(rh'(r)+h(r)^2-h(r)\right)\right\} \ .
\end{aligned}
\tag{1.35}
$$

In vacuum, $T_{\mu\nu} = 0$ which leads to the Einstein field equation (1.22) as

$$
R = 0 \qquad \text{and} \qquad R_{\mu\nu} = 0 \tag{1.36}
$$

Using the above equation ($R_{tt} = 0 = R_{rr}$) for the metric (1.33), we find the following equation after simplification

$$
h(r)f'(r) + f(r)h'(r) = 0 \ . \tag{1.37}
$$

The solution of the above equation gives the relation $h(r) = \frac{1}{f(r)}$ which turn out as $g_{tt} = \frac{1}{g_{rr}}$. This relation, eq.(1.36) ($R = 0$) and eq.(1.35) give us

$$
R = -\frac{1}{r^2}\left(-2+2f(r)+4rf'(r)+r^2f''(r)\right) = 0 \tag{1.38}
$$

which is a single equation for $f(r)$. The solution of the above equation is obtained as

$$
f(r) = 1 + \frac{const.}{r} \tag{1.39}
$$

Using Newtonian gravity limit, one can obtain this constant as $const. = -\frac{2GM}{c^2}$ where M is the source-mass. This leads us to obtain the metric solution

$$
ds^2 = -(1-\frac{2GM}{c^2r})dt^2 + \frac{dr^2}{(1-\frac{2GM}{c^2r})} + r^2d\Omega_2^2 \tag{1.40}
$$

which is known as Schwarzschild black hole solution where M is the mass of the black hole. The eq.(1.40) represents the Schwarzschild black hole with horizon radius $r_h = 2GM$ which is a coordinate singular point of the metric. The role of space and time exchange when $r < r_h$ and this special property helps black hole to become black because nothing can come out from black hole. The horizon surface is the boundary surface in spacetime beyond which events cannot influence an outside observer. That's why this region is called black hole.

1.3 Black hole thermodynamics

The horizon area is given by [50]

$$A = 4\pi r_h^2 = 16\pi G^2 M^2 \tag{1.41}$$

which indicates that horizon area increases if matter falls in the black hole. Since nothing comes out from black hole, the area is a non-decreasing quantity which resembles of thermodynamic entropy. This is the first indication that black hole is a thermodynamic object.
From **No Hair Theorem** [50],[51] we know that a stationary asymptotically flat black hole is fully characterized by mass M, angular momentum J and charge Q of the black hole. The properties of black hole does not contain any properties of original stars. On the other hand, the properties of a thermodynamic object does not depend on the position or momentum of molecules. Only few quantities (macroscopic variable) is needed to specify a black hole. This is also another indication that black hole is a thermodynamic object. To protect the second law of thermodynamics for universe, Bekenstein [52] first proposed that black hole have entropy which is proportional to horizon surface area ($S_{BH} \propto A$). The entropy of black hole is given by [52],[53]

$$S = \frac{A}{4G\hbar} \ . \tag{1.42}$$

The entropy counts the number of possible quantum states when macroscopic variables are specified. So the entropy is proportional to the number of particles ($S \propto N$). The black hole entropy counts the microscopic states of a black hole. Consider a black hole made of large number of particles in small region. It is not possible to collect a particle with mass m in a region which is smaller than Compton wavelength $\lambda = \frac{\hbar}{m}$. To form a black hole from this particle, λ should less than the Schwarzschild radius $r_h = 2GM$ where M is the mass of black hole. The lightest particle must satisfies

$$\frac{\hbar}{m} = 2GM \quad \Rightarrow m = \frac{\hbar}{2GM} \tag{1.43}$$

The total number of particle inside the black hole is given by

$$N_{max} = \frac{M}{m} = \frac{2G^2M^2}{G\hbar} = \frac{A}{8\pi G\hbar} \tag{1.44}$$

which matches with order of magnitude with exact result of entropy of black hole. This is the heuristic way to describe the origin of black hole entropy. We now want to calculate some thermodynamics quantities for a black hole.

1.3.1 Surface gravity and Hawking temperature

$\boxed{Surface\ gravity}$: Surface gravity is the acceleration of stationary observer at horizon which is measured at infinity. The local acceleration is defined as

$$a = \frac{1}{\rho} \tag{1.45}$$

where ρ is the Rindler coordinate. The relation between the energy of an object at different points A and B is related by $\frac{E_A}{E_B} = \sqrt{\frac{g_{00(B)}}{g_{00(A)}}}$ which can be shown from metric. This implies that the acceleration measured at infinity is related by

$$a_\infty = \sqrt{\frac{g_{00}(r)}{g_{00}(\infty)}}\, a \tag{1.46}$$

A spherical symmetric and static black hole is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{1.47}$$

where $f(r) = 1 - \frac{r_h}{r}$ and $r_h = 2GM$. To know the gravity at the horizon surface, we need to expand $f(r)$ about horizon

$$\begin{aligned} f(r) &= f(r_h) + (r - r_h)f'(r_h) + \ldots\ldots\ldots \\ f(r) &\approx (r - r_h)f'(r_h) \end{aligned} \tag{1.48}$$

It is not possible to access beyond horizon. So we need to define proper distance in which horizon is the starting point of distance. We define proper distance as

$$d\rho = \frac{dr}{\sqrt{f(r)}} = \frac{dr}{\sqrt{(r - r_h)f'(r_h)}} \tag{1.49}$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_h)}}\sqrt{r - r_h}\ . \tag{1.50}$$

Defining $\eta = \frac{f'(r_h)}{2}t$ and substituting value of ρ in $f(r)$, we get near horizon black hole geometry

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + r_h^2 d\Omega_2^2 \tag{1.51}$$

which is Rindler $\times S^2$ geometry and (η, ρ) is called Rindler coordinate. From definition of local acceleration, we find

$$a(r) = \frac{1}{\rho} = \frac{\sqrt{f'(r_h)}}{2\sqrt{r - r_h}}\ . \tag{1.52}$$

From definition of surface gravity and eq.(1.46), we get

$$\begin{aligned} \tilde{\kappa} := a_\infty &= a(r)\sqrt{(r - r_h)f'(r_h)} = \frac{\sqrt{f'(r_h)}}{2\sqrt{r - r_h}}\sqrt{(r - r_h)f'(r_h)} \\ \Rightarrow \tilde{\kappa} &= \frac{f'(r_h)}{2} \end{aligned} \tag{1.53}$$

which is the measure of surface gravity.

Hawking Temperature : The matter outside of black hole with quantum description allows thermal radiation from horizon surface. Hawking [54] showed that black hole which was formed due to gravitational collapse, acts like blackbody with finite temperature. There are many ways to derive the Hawking temperature. The Euclidean formalism is the simplest way to derive Hawking temperature. A general black hole geometry in any arbitrary dimension is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2 \tag{1.54}$$

where $f(r_h) = 0 = g(r_h)$ and last part is angular part. To study any quantum field theory (QFT) at finite temperature, we change time to Euclidean time ($t \to i\tau$) in which periodic identification of τ leads to the temperature T of the QFT. The period of τ is given by

$$\tau \to \tau + \frac{\hbar}{T} \ . \tag{1.55}$$

Changing time to Euclidean time, the metric (1.54) reads

$$ds^2 = +f(r)d\tau^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2 \ . \tag{1.56}$$

Taylor expansion of $f(r)$ and $g(r)$ about horizon are give respectively

$$\begin{aligned} f(r) &= f(r_h) + (r - r_h)f'(r_h) + \ldots\ldots\ldots \\ &\approx (r - r_h)f'(r_h) \end{aligned} \tag{1.57}$$

$$\begin{aligned} g(r) &= g(r_h) + (r - r_h)g'(r_h) + \ldots\ldots\ldots \\ &\approx (r - r_h)g'(r_h) \end{aligned} \tag{1.58}$$

Beyond horizon is forbidden spacetime, so we define proper distance from horizon as

$$d\rho = \frac{dr}{\sqrt{g(r)}} = \frac{dr}{\sqrt{g'(r_h)(r - r_h)}}$$

$$\rho = \frac{2}{\sqrt{g'(r_h)}}\sqrt{r - r_h} \tag{1.59}$$

$$(r - r_h) = \frac{\rho^2}{4} g'(r_h) \ . \tag{1.60}$$

The black hole geometry near horizon reads

$$\begin{aligned} ds^2 &= (r - r_h)f'(r_h)d\tau^2 + d\rho^2 + r_h^2 d\Omega^2 \\ &= \frac{\rho^2}{4} g'(r_h)f'(r_h)d\tau^2 + d\rho^2 + r_h^2 d\Omega^2 \end{aligned} \tag{1.61}$$

$$= \rho^2 d\theta^2 + d\rho^2 + r_h^2 d\Omega^2 \tag{1.62}$$

where $\theta = \frac{\sqrt{g'(r_h)f'(r_h)}}{2}\tau$. This is Euclidean flat space in polar coordinate. To eliminate conical singularity at $\rho = 0$, the θ coordinate have to period of 2π which reads

$$\begin{aligned} \theta &\to \theta + 2\pi \\ \Rightarrow \tau &\to \tau + \frac{4\pi}{\sqrt{g'(r_h)f'(r_h)}} \end{aligned} \tag{1.63}$$

Compare eq.(1.63) and eq.(1.55), we find

$$T = \frac{\hbar}{4\pi}\sqrt{g'(r_h)f'(r_h)} \tag{1.64}$$

which is called Hawking temperature T_H. For Schwarzschild black hole, we know that $f(r) = g(r)$ which leads to Hawking temperature

$$\begin{aligned} T_H &= \frac{\hbar}{4\pi}f'(r_h) && (1.65) \\ T_H &= \frac{\hbar\tilde{\kappa}}{2\pi} && (1.66) \end{aligned}$$

where $\tilde{\kappa} = \frac{1}{2r_h}$ is the surface gravity for Schwarzschild black hole. This tells us that surface gravity plays crucial role for Hawking temperature.

1.3.2 Thermodynamic quantities for black hole

From first law of thermodynamics $dE = TdS$ and $E = M$ for black hole, we find

$$\begin{aligned} \frac{dS_{BH}}{dE} &= \frac{1}{T_H} = \frac{4\pi r_h}{\hbar} = \frac{8\pi GM}{\hbar} && (1.67) \\ S_{BH} &= \frac{4\pi GM^2}{\hbar} = \frac{A}{4\hbar G} && (1.68) \end{aligned}$$

where A is the area of horizon surface. This is different from the statistical entropy of non-gravitational object which is proportional to volume of the system. The area of 5-dimensional spacetime is the 4-dimensional volume. This is the **first clue** of holographic theory in which one has to consider one higher spatial dimensional gravity theory to describe a phenomena of a non-gravitational system which lives on the boundary of higher dimensional spacetime. Then we can argue that degree of freedom of gravitational theory in bulk and degree of freedom of non-gravitational system in boundary are same because the degree of freedom of gravitational theory is counted at boundary of that spacetime. The specific heat of Schwarzschild black hole is given by

$$C_{BH} = \frac{\partial E}{\partial T_H} = -\frac{8\pi GM^2}{\hbar} < 0 \ . \tag{1.69}$$

Generally, gravitational system has negative specific heat. But we know that the specific heat of any statistical system is always positive. To describe a phenomena of a non-gravitational system by using a gravitational theory, one has to find a black hole geometry which has positive specific heat. The AdS-Schwarzschild black hole with planner symmetry always has positive specific heat. This is **a clue** for considering AdS spacetime in gauge/gravity duality.
Free energy of a black hole is given by

$$F = E - TS \ . \tag{1.70}$$

For Schwarzschild black hole, we know that $E = M$, $T = \frac{\hbar}{8\pi GM}$ and $S = \frac{4\pi GM^2}{\hbar}$. Free energy of this black hole reads

$$F = M - \frac{M}{2} = \frac{M}{2} \ . \tag{1.71}$$

The Euler relation in thermodynamics ($E = TS$) does remain valid in black hole thermodynamics. For this black hole we get

$$E = 2TS \tag{1.72}$$

which is called Smarr's formula. The reason behind this formula is that entropy ($S_{BH} \propto E^2$) is a second order function of E whereas entropy in thermodynamic system is first order function of E (extensive variables).

1.3.3 Laws of black hole thermodynamics

Zeroth law : It states that surface gravity is constant over the horizon in thermal equilibrium.
This is similar as zeroth law of thermodynamics which tells us that temperature of a thermal equilibrium system is constant everywhere. The horizon of a stationary black hole always constant which implies equilibrium state of black hole and constant surface gravity.
First law : It states that energy is conserved.

$$dM = \frac{\tilde{\kappa}}{8\pi G} dA + \Omega dJ + \phi dQ \tag{1.73}$$

where M is mass of black hole, $\tilde{\kappa}$ is surface gravity, A is horizon area, Ω is angular velocity, J is angular momentum, ϕ is electric potential and Q is electric charge. This law is also similar to 2nd law of thermodynamics which is

$$\begin{aligned} dE &= dU + dW \\ dE &= TdS \end{aligned} \tag{1.74}$$

where U and W are internal energy and work done respectively. For Schwarzschild black hole ($Q = 0, J = 0$), the horizon area A increases if mass M of the black hole increases from eq.(1.41). This implies

$$dM = \frac{dA}{32\pi G^2 M} = \frac{\tilde{\kappa}}{8\pi G} dA \tag{1.75}$$

The energy of this black hole is mass of the the black hole and $S \propto A$. So we compare eq.(1.74) with eq.(1.75), we must say that black hole temperature T_H is related to surface gravity of the black hole.
Second law : Total entropy of a system consisting of black hole and matter never decreases.

$$dS_{tot} = dS_{matter} + dS_{BH} \geqslant 0 \tag{1.76}$$

Third law (Nernst's law) : It is impossible to reduce the surface gravity to zero by a finite sequence of operations.
This discussion will help us to understand the basic structure of gauge/gravity duality. In next section, we will discuss briefly about non-linear generalization of Maxwell electrodynamics which is known as "Born-Infeld (BI) electrodynamics".

1.4 Born-Infeld Electrodynamics

In [55], the ten biggest problems in physics have been discussed in the present day. The possible answer may be lurk in the non-linearity of a theory. There are many theoretical aspects as well as experimental aspects for generalization of non-linear version of any known linear theory. The Born-Infeld theory was developed for purely theoretical purpose. However the result of the experiment of multiphoton light-by-light scattering [56] showed that electrodynamics in a vacuum constitutes a nonlinear theory.

Starting with Maxwell theory of electromagnetism which is linear theory, we interpret the relation between matter and electromagnetic field as the charged particle which are the source of the field, not a part of the field. The characteristic property of charged particle is inertia which is measured by the mass. When we calculate the self-energy of the charged particle due to electrostatic repulsion, it diverges. The electric field or electric potential energy diverge on the surface of a charged point particle. To remove this divergence, Gustav Mie [57] first generalized the Maxwell's theory in non-linear form. Later on Born [58] proposed a non-linear electromagnetic theory in 1932, which exhibited vacuum birefringence. In the same year Born and Infeld [59] modified the Born theory which did not exhibit vacuum birefringence.

Underlying Lagrangian density for Maxwell's theory is $\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}(B^2 - E^2)$ which is replaced by [58]

$$\mathcal{L}_B = b^2\left(1 - \sqrt{1 + \frac{1}{2b^2}F^{\mu\nu}F_{\mu\nu}}\right) \quad \Rightarrow \quad \mathcal{L}_B = b^2\left(1 - \sqrt{1 + \frac{1}{b^2}(B^2 - E^2)}\right) \tag{1.77}$$

To avoid letting physical quantity become infinite, there should be upper limit of velocity of a particle. If we impose an upper limit of velocity of a particle, say, velocity of light c, then Newtonian Lagrangian of a free particle $L = (\frac{1}{2}mv^2)$ is replaced by

$$L = mc^2\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \quad \Rightarrow \quad L = a^2\left(1 - \sqrt{1 - \frac{mv^2}{a^2}}\right) \tag{1.78}$$

where $a^2 = mc^2$. Using this same analogy EM field which also has an upper limit of the field strength which modifies $\mathcal{L}_M$ to $\mathcal{L}_B$. In the limit of $b \to \infty$, we will recover Maxwell's electrodynamics. Now we try to introduce the Lagrangian density using some physical argument, which has been introduced by Born and Infeld. There are two Lorentz invariant quantities in electromagnetic theory. All this case we have only considered the invariant term $(B^2 - E^2)$. To consider both Lorentz invariant quantities in the Lagrangian, we need to find out general way to compute this BI Lagrangian. First consider a general covariant tensor field a_{kl} and the Lagrangian density is function of a_{kl} in such way that invariance of action leads

$$\mathcal{L}_{BI} = \sqrt{|a_{kl}|} \tag{1.79}$$

Any arbitrary tensor a_{kl} can be split up into a symmetrical and anti-symmetrical part. Therefore, we can write

$$a_{kl} = g_{kl} + f_{kl} \tag{1.80}$$

where $g_{kl} = g_{lk}$ and $f_{kl} = -f_{lk}$. These two may be interpreted as metrical field and electromagnetic field respectively. For our convention, we need one negative sign inside the square root. So the metric can be cast in following possible way

$$\sqrt{-|a_{kl}|} = \sqrt{-|g_{kl} + f_{kl}|} \quad ; \quad \sqrt{-|g_{kl}|} \quad ; \quad \sqrt{|f_{kl}|} \tag{1.81}$$

The simplest form of $\mathcal{L}_{BI}$ is linear combination of three possible cases, which reads

$$\mathcal{L}_{BI} = -\sqrt{-|g_{kl} + f_{kl}|} + A\sqrt{-|g_{kl}|} + B\sqrt{|f_{kl}|} \tag{1.82}$$

The negative sign in first term at RHS of above equation is due to sign convention. The space-time integral of anti-symmetric tensor can be changed into surface integral which has no influence on variational equation. This implies $B = 0$. To determine A, we consider small value of f_{kl} and cartesian coordinate system. We know that Lagrangian density of Maxwell' theory can be written in terms of antisymmetric tensor which can be identified with our general antisymmetric tensor f_{kl} and the Lagrangian density of Maxwell electrodynamics becomes

$$\mathcal{L}_M = -\frac{1}{4} f^{kl} f_{kl} = \frac{1}{2}\left(f_{12}^2 + f_{13}^2 + f_{14}^2 - f_{23}^2 - f_{42}^2 - f_{34}^2\right) \ . \tag{1.83}$$

In this coordinate system $g_{kl} = \delta_{kl}$ and the determinant reads

$$-|\delta_{kl} + f_{kl}| = 1 - \left(f_{12}^2 + f_{13}^2 + f_{14}^2 - f_{23}^2 - f_{42}^2 - f_{34}^2\right) - \left(f_{12}f_{34} + f_{13}f_{42} + f_{23}f_{14}\right)^2 \ . \tag{1.84}$$

We can neglect last term in the above equation since we assume small value of f_{kl}. With this assumption and binomial expansion, the above equation recast as

$$-\sqrt{-|\delta_{kl} + f_{kl}|} \approx -1 + \frac{1}{2}\left(f_{12}^2 + f_{13}^2 + f_{14}^2 - f_{23}^2 - f_{42}^2 - f_{34}^2\right) \tag{1.85}$$

Compare eq.(s)(1.82,1.83,1.85), we find $A = 1$. Therefore the BI Lagrangian reads

$$\mathcal{L}_{BI} = \sqrt{-|g_{kl}|} - \sqrt{-|g_{kl} + f_{kl}|} \tag{1.86}$$

which translate in Cartesian coordinate

$$\mathcal{L}_{BI} = 1 - \sqrt{1 + F - G^2} \tag{1.87}$$

where $F = -(f_{12}^2 + f_{13}^2 + f_{14}^2 - f_{23}^2 - f_{42}^2 - f_{34}^2)$ and $G = f_{12}f_{34} + f_{13}f_{42} + f_{23}f_{14}$. We define $|g_{kl}| = g$ and the determinant reads in general coordinate

$$\mathcal{L}_{BI} = \sqrt{-g}\left(1 - \sqrt{1 + F - G^2}\right) \ . \tag{1.88}$$

Using Maxwell theory we can identify

$$F = (B^2 - E^2) = \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \tag{1.89}$$

$$G = -\vec{E}.\vec{B} = \frac{1}{4} F^{\mu\nu} F^*_{\mu\nu} \tag{1.90}$$

where $F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is dual field tensor and $\epsilon_{\mu\nu\alpha\beta}$ is Levi-Civita tensor. The maximum value of field strength is denoted by b which has dimension of field strength. Using dimension analysis and limiting case ($b \to \infty$), we find the corrected form of the BI Lagrangian which reads [59], [60]

$$\begin{aligned}\mathcal{L}_{BI} &= b^2\sqrt{-g}\left(1-\sqrt{1+\frac{F}{b^2}-\frac{G^2}{b^4}}\right)\\ &= b^2\sqrt{-g}\left(1-\sqrt{1+\frac{F^{\mu\nu}F_{\mu\nu}}{2b^2}-\frac{(F^{\mu\nu}F^*_{\mu\nu})^2}{16b^4}}\right) \ . \qquad (1.91)\end{aligned}$$

This Lagrangian density contains two Lorentz invariant quantities and it have electromagnetic duality ($E \leftrightarrow -B,\ B \leftrightarrow E$). To incorporate this electrodynamics in a medium (macroscopic body), we introduce another second rank anti-symmetric field tensor p_{kl} which is defined as

$$\sqrt{-g}p^{kl} = -\frac{\partial \mathcal{L}}{\partial f_{kl}} \ . \qquad (1.92)$$

In Cartesian coordinate, we get

$$H = -\frac{\partial \mathcal{L}_{BI}}{\partial B} = \frac{\vec{B} - G\vec{E}}{\sqrt{1+\frac{F}{b^2}-\frac{G^2}{b^4}}} \qquad (1.93)$$

$$D = -\frac{\partial \mathcal{L}_{BI}}{\partial E} = \frac{\vec{E} - G\vec{B}}{\sqrt{1+\frac{F}{b^2}-\frac{G^2}{b^4}}} \ . \qquad (1.94)$$

If we match this result with Maxwell theory in a medium (macroscopic body), the H and D are identified as magnetizing field and displacement field respectively. In a medium, the field equations becomes

$$\begin{aligned}\vec{\nabla}.\vec{B} &= 0\\ \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0\\ \vec{\nabla}.\vec{D} &= 0\\ \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} &= 0 \qquad (1.95)\end{aligned}$$

For electrostatic field of a point charge, the E and D are independent of time and $B = 0$ and $H = 0$. From the above field equations, we get for spherical symmetry

$$\vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{E}_r = -\frac{d\phi}{dr} \qquad (1.96)$$

$$\vec{\nabla}.\vec{D} = 0 \quad \Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = 0 \qquad (1.97)$$

where ϕ is electrostatic potential. The solution of eq.(1.97) is

$$D_r = \frac{e}{r^2} \qquad (1.98)$$

where e is the charge of point particle (electron). From definition of displacement field we find from eq.(1.94)

$$D_r = \frac{E_r}{\sqrt{1 - \frac{E_r^2}{b^2}}} \ . \tag{1.99}$$

Using eq.(1.96) and eq.(1.98), we find

$$\begin{aligned}
\frac{e}{r^2} &= \frac{-\frac{d\phi}{dr}}{\sqrt{1 - \frac{1}{b^2}(\frac{d\phi}{dr})^2}} \\
\Rightarrow \frac{d\phi(r)}{dr} &= \frac{e}{r_0^2} \frac{dr}{\sqrt{1 + \left(\frac{r}{r_0}\right)^4}} \\
\phi(r) &= \frac{e}{r_0} \int_x^{\infty} \frac{dy}{\sqrt{1 + y^4}} \equiv \frac{e}{r_0} f(r/r_0) \qquad (1.100)
\end{aligned}$$

where $r_0^2 = \frac{e}{b}$, $y = \frac{r}{r_0}$ and $f(x) = \int_x^{\infty} \frac{dy}{\sqrt{1+y^4}}$. This is the elementary potential for a point charge. Substituting $y = tan\frac{\beta}{2}$ in $f(x)$, we get

$$f(x) = \frac{1}{2} \int_{\beta'=2tan^{-1}(x)}^{\pi} \frac{d\beta}{\sqrt{1 - \frac{1}{2}sin^2(\beta)}} = f(0) - \frac{1}{2} F\left(\frac{1}{\sqrt{2}}, \beta'\right) \tag{1.101}$$

where $F(a, b) = \int_0^a \frac{dt}{\sqrt{1-b^2 sin^2(t)}}$ is elliptic function. In eq.(1.101) we substitute $x = 0$ which leads to $\beta' = 0$ and

$$\begin{aligned}
f(0) - \frac{1}{2} F\left(\frac{1}{\sqrt{2}}, 0\right) &= \frac{1}{2} \int_0^{\pi} \frac{d\beta}{\sqrt{1 - \frac{1}{2}sin^2(\beta)}} \\
\Rightarrow f(0) &= \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - \frac{1}{2}sin^2(\beta)}} \equiv F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) \\
\Rightarrow f(0) &= 1.8541 \qquad (1.102)
\end{aligned}$$

So the potential at origin ($r = 0$) reads

$$\phi(0) = \frac{e}{r_0} f(0) = 1.8541\sqrt{e}\sqrt{b} \tag{1.103}$$

From eq.(1.99) we finally get

$$E_r(r) = \frac{e}{r_0^2 \sqrt{1 + (\frac{r}{r_0})^4}} \tag{1.104}$$

From eq.(1.100) and eq.(1.104), we must say that electric field and potential are finite at origin ($r = 0$) for point charge e and these values are

$$\begin{aligned}
\phi(0) &= 1.8541\sqrt{eb} \\
E_r(0) &= b \ . \qquad (1.105)
\end{aligned}$$

This implies that b is the maximum field strength which is measured at origin of the field. To determine the value of b, we first calculate the energy-momentum tensor which gives the energy density of this theory in following way

$$T^{00} = 4\pi u = \vec{D}.\vec{E} - \mathcal{L}_{BI} \ . \tag{1.106}$$

In electrostatic case, the total energy of the charged particle reads

$$\begin{aligned} U &= \int u dv = \int_0^\infty 4\pi r^2 u dr \\ &= \int_0^\infty dr \left[\frac{e^2}{r_0^2\sqrt{1+(\frac{r}{r_0})^4}} - \frac{e^2}{r_0^4} r^2 \left(1 - \frac{\frac{r^2}{r_0^2}}{\sqrt{1+\frac{r^4}{r_0^4}}} \right) \right] \\ &= (1.8541 - 0.6180)\frac{e^2}{r_0} \\ &= 1.2361 \frac{e^2}{r_0} \end{aligned} \tag{1.107}$$

But rest energy of static charge particle (electron) is $U = m_0 c^2$ which is compared with eq.(1.107), we get

$$r_0 = 1.2361 \frac{e^2}{m_0 c^2} = 2.28 \times 10^{-13} e.s.u. \tag{1.108}$$

$$\Rightarrow b = \frac{e}{r_0^2} = 9.18 \times 10^{15} e.s.u. \tag{1.109}$$

So the maximum value of field strength is very high. This values indicates that Born-Infeld electrodynamics plays crucial role to know inner structure of electron. We get the main goal of Born-Infeld theory, which is to obtain a point-like charge solution with finite self-energy. Considering an absolute field b which is an upper limit of a purely electric field, we can derive all field equations and their properties. This gives us generalized non-linear version of Maxwell theory in which we remove divergence of self energy of point like charge particle. It has a application in cosmic (high frequency) rays. This type of generalization comes spontaneously in different theories, namely gravity theory, string theory. One of the predictions of string theory is that the low-energy effective Lagrangian describing electromagnetism should be of the Born–Infeld (BI) type [61]-[63]. There is an experimental proposal to measure the value of $b^2 \sim 10^{32} G$ using laser ring in [64] and a test in Waveguides [65]. This theory is difficult to quantize since it is a non-linear theory. Dirac had tried to quantize it by constraints analysis, but he failed [60]. There are no general way to quantize a non-linear theory. There are several application of BI electrodynamics such as correction to Coulombian interactions [66],[67], BI electrodynamics on a lattice [68], application in astrophysics [69] etc. In the context of the application of gauge/gravity duality, this electrodynamics plays crucial role in phase transition of gravity theory.

1.5 Overview of the thesis

The physics of strongly coupled systems poses difficulties when approached by conventional methods. The gauge/gravity duality provides a powerful mathematical tool to study strongly coupled field theoretical systems by investigating weakly coupled gravitational systems. Using black hole phase transition, we can describe phase transitions in strongly coupled systems with the help of this AdS/CFT dictionary.

It is not easy task to find out any phase transition in gravity side. Hawking and Page [70] first showed that there is a first order phase transition in gravity theory. They showed that a AdS-Schwarzschild black hole geometry changes to thermal AdS spacetime. The physical interpretation is that black hole evaporates and can shrink completely to photon gas by Hawking radiation. There is a discontinuity in entropy when the AdS-Schwarzschild black hole changes to thermal AdS spacetime since thermal AdS does not have any entropy. This tells us that Hawking-Page phase transition is a first order phase transition. The dual theory of this phase transition is associated with confinement/de-confinement phase transition in QCD [9].

For second order phase transition, the change of entropy is continuous which tells us that we need to find a phase transition from one black hole geometry to another black hole geometry. It is not trivial because of no hair theorem. Gubser [33] first showed that the scalar hair formation outside a black hole is possible for asymptotic AdS spacetime. This provides us one clue of second order phase transition in which phase transition occurs from a black hole geometry with scalar hair to black hole geometry without scalar hair. The dual theory of this phase transition is associated with phase transition in strongly coupled superconductors [71]-[76].

Using the AdS/CFT duality, the lower bound of shear viscosity for strongly coupled $\mathcal{N} = 4$ SYM plasma has been explained in [14]. The computation of the entanglement entropy for a quantum field theory was developed in [77]-[79]. From gravity model, we can also compute entanglement entropy using minimal area prescription in [22] which is called holographic entanglement entropy(HEE). The HEE agrees with the results obtained from the field theory side. The details of the chapters of this thesis are as follows.

The formalism of gauge/gravity will be discussed in the next chapter. This is a short review of gauge/gravity without detailed calculations in the original paper [4]. We will first discuss about basic structure of Anti-de Sitter (AdS) spacetime and the special properties of this spacetime. We also will show how the symmetry of this spacetime leads to a lower dimensional conformal field theory (CFT). We then discuss briefly about the symmetry of CFT. A bit of string theory and D-brane physics will then be discussed to understand the AdS/CFT correspondence. The low energy limit of D3-brane physics leads to $AdS_5 \times S^5$ spacetime which is equivalent (symmetric wise) to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensional spacetime. We have also showed the connection between gauge theory and string theory. From string theory, we are able to establish the duality between gauge theory (CFT) and gravity theory (AdS). The GKPW formalism have been discussed in the next section. Using GKPW prescription, we will establish the AdS/CFT dictionary which is very essential for the thesis. We then move to show the application of this duality in understanding phase transitions in different physical systems.

In Chapter 3, we have analytically investigated holographic superconductors in presence of non-linear electrodynamics in the background of Gauss-Bonnet gravity in arbitrary spacetime dimension which is based on our work [35]. Before presenting our work, we have first discussed the theory of superconductivity in details. We then discuss about the basic idea of holographic superconductors. Using the Sturm-Liouville (SL) eigenvalue method, we have presented our first work on holographic superconductors in the presence of Born-Infeld electrodynamics in background of GB gravity with backreaction. The main motivation of this work is to know whether all these parameters (BI parameter, GB parameter, backreaction parameter and spacetime dimension) favour or disfavour for condensation in holographic superconductors. We have calculated the critical temperature and the condensation operator value which depend on these parameters. The higher values of these parameters disfavour the formation of the condensate. Our analytical results agree with the numerical findings in the literature.

In Chapter 4, the holographic free energy and thermodynamic geometry of holographic superconductor in the presence of Maxwell electrodynamics and Born-Infeld electrodynamics have been investigated analytically. This chapter is based on our two published works [36, 37]. Using the matching method, we have first calculated the critical temperature and the condensation operator value. In the matching method, we have matched two behaviours (horizon behaviour and asymptotic behaviour) of fields at any arbitrary point between the horizon and the boundary. We then investigate the free energy of holographic superconductors in the presence of linear (Maxwell) and non-linear (BI) electrodynamics. From the investigation of holographic free energy, we are able to use Ruppeiner formalism for studying the thermodynamic geometry of the system. From this geometry, one can calculate Ruppeiner scalar curvature which carries information about the system. The divergence of the Ruppeiner scalar curvature indicates the critical phenomena of the system. This will lead us to know the critical temperature of the holographic superconductor. The result from the divergence of the scalar curvature match with other analytical methods, namely, matching method, Sturm-Liouville eigenvalue method as well as numerical findings in the literature.

In Chapter 5, we have analytically obtained the conductivity of holographic superconductor in presence of Born-Infeld electrodynamics [38]. We have first discussed the Drude model of conductivity which will help us to understand the conductivity of holographic superconductors. From the investigation of dc conductivity, one can estimate the ratio between the energy gap and the critical temperature. To incorporate the BI parameter in the energy gap, we have first calculated the backreacted metric which depends on the BI parameter. Using this backreacted metric, we have analytically investigated conductivity of holographic superconductor in presence of

BI electrodynamics. This investigation also help us to know the ratio of energy gap and the critical temperature. In the Maxwell limit, we have recovered the result which agree with numerical finding as well as experimental finding. Then we carry out the computation of conductivity using a self-consistent approach. In self-consistent method, we have again shown that the dc conductivity of holographic superconductor is infinite. With the backreaction parameter, it is observed that the ratio of the energy gap and the critical temperature is increasing with higher value of the BI parameter.

In Chapter 6, the response of holographic superconductor to the external magnetic field has been investigated in the presence of Born-Infeld electrodynamics. This chapter is based on our work [39]. Starting with discussion on Meissner effect of superconductor, we proceed with holographic superconductor model. Using the matching method, the critical temperature and the condensation operator value has been analyzed. In the presence of external magnetic field, the equation of motion of our holographic system has been solved by using the matching method. This solution leads us to find the critical magnetic field of our holographic superconductor model in presence of BI electrodynamics. In this analysis, we have shown that the critical magnetic field increases with higher value of the BI parameter.

We have summarized our findings in Chapter 7. From the basic pillar of gauge/gravity duality, the structure of this duality and the application of this duality to analyze strongly coupled systems have been discussed along with our original work in this thesis. We have summarized our findings and motivation of our works. The future work in this direction has been also discussed in this chapter.

Chapter 2

AdS/CFT correspondence

String theory [80]-[83] provides a framework to incorporate finite quantum corrections to classical gravity. It proposes that the constituent of matter is made up very tiny strings instead of point-particle. All interactions are supposed to occur by splitting and joining of strings which lives in 10 dimensional spacetime. To describe four dimensional real world, it is considered that six of the dimensions are compact and very tiny. All elementary particles are just different modes of string oscillation. There are two type of strings, namely, open string and closed string. D-brane are described as surfaces on which open string end [84]. Using D-brane physics, it was shown in [4] that $\mathcal{N} = 4$ super Yang-Mills (SYM) $SU(N)$ gauge theory is dual to classical supergravity theory in $AdS_5 \times S^5$. It carries different names, AdS/CFT correspondence, gauge/string duality, gauge/gravity duality, weak/strong duality, bulk/boundary duality. Before going to details of this duality, we start with discussion of some basic ingredient of this duality, namely, Anti-de Sitter spacetime (AdS), conformal field theory (CFT), string theory and D-brane physics and Holographic principle.

2.1 Anti-de Sitter Spacetime

The Anti-de Sitter spacetime is a maximally symmetric spacetime with a negative constant curvature. Some parts of this section is based upon the discussion in [85]. To know the structure of this spacetime, we start with space having constant curvature.

2.1.1 Space with constant curvature

$\boxed{S^2}$: A sphere is defined by the following condition

$$X^2 + Y^2 + Z^2 = L^2 \tag{2.1}$$

where L is the radius of the sphere. Here, the line element is defined by

$$ds^2 = dX^2 + dY^2 + dZ^2 \tag{2.2}$$

which is 3- dimensional Euclidean space metric. Changing the variables

$$X = L\sin\theta\cos\phi,\ Y = L\sin\theta\sin\phi,\ Z = L\cos\theta$$

which satisfies eq.(2.1), we get the metric in the following form

$$ds^2 = L^2\left[d\theta^2 + \sin^2\theta d\phi^2\right] \tag{2.3}$$

which has $SO(3)$ invariance. This also has a constant positive curvature (Ricci scalar $R = \frac{2}{L^2}$).

$\boxed{H^2}$: We now consider hyperbolic space H^2 which has negative curvature, is defined as

$$-Z^2 + X^2 + Y^2 = -L^2\ . \tag{2.4}$$

This space cannot be embedded into 3-dimensional Euclidean space. It is embedded in 3-dimensional Minkowski space which is defined by the following metric

$$ds^2 = -dZ^2 + dX^2 + dY^2\ . \tag{2.5}$$

Using new variables

$$X = L\sinh\rho\cos\phi,\ Y = L\sinh\rho\sin\phi,\ Z = L\cosh\rho$$

which satisfies eq.(2.4), we get the metric

$$ds^2 = L^2\left[d\rho^2 + \sinh^2\rho d\phi^2\right] \tag{2.6}$$

This has $SO(1,2)$ invariance which is associated with Lorentz transformation. The Ricci scalar for this metric is $R = -\frac{2}{L^2}$. Now we move to spacetime with constant curvature.

2.1.2 Spacetime with constant curvature

$\boxed{dS_2}$: de Sitter spacetime is embedded into flat spacetime with one timelike direction. This spacetime is defined as

$$-Z^2 + X^2 + Y^2 = L^2 \tag{2.7}$$

where L is the radius of the sphere. Here, the line element is defined by

$$ds^2 = -dZ^2 + dX^2 + dY^2\ . \tag{2.8}$$

We change the coordinate system

$$X = L\cosh\tilde{t}\cos\theta,\ Y = L\cosh\tilde{t}\sin\theta,\ Z = L\sinh\tilde{t}$$

using eq.(2.7), we obtain

$$ds^2 = L^2\left[-d\tilde{t}^2 + \cosh^2\tilde{t}\ d\theta^2\right] \tag{2.9}$$

which has $R = \frac{2}{L^2}$ and $SO(1,2)$ invariance. This spacetime has huge applications in cosmology.
$\boxed{AdS_2}$: Anti-de Sitter spacetime can be embedded into flat spacetime with two timelike direction. The line element is defined as

$$ds^2 = -dZ^2 - dX^2 + dY^2 \tag{2.10}$$

for the spacetime which satisfies the constraint

$$-Z^2 - X^2 + Y^2 = -L^2 \ . \tag{2.11}$$

This spacetime has $SO(2,1)$ invariance. Reparametrizing the coordinate system

$$X = L\cosh\rho\sin\tilde{t}, \ Y = L\sinh\rho, \ Z = L\cosh\rho\cos\tilde{t}$$

we find

$$ds^2 = L^2\left[-\cosh^2\rho \ d\tilde{t}^2 + d\rho^2\right] \tag{2.12}$$

where $(\tilde{t},\rho)$ is called global coordinate for AdS_2. Although the embedded spacetime has two timelike direction, this spacetime itself has only one timelike direction. The reparametrized coordinate shows that $\tilde{t}$ has periodicity 2π and $\rho \in [-\infty,+\infty]$. This is problematic, so one should take the universal cover $\tilde{t} \in \mathcal{R}$. Taking $r = L\sinh\rho$ and $t = L\tilde{t}$, we can write metric in following form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} \tag{2.13}$$

where $f(r) = 1 + \frac{r^2}{L^2}$ and $t \in [-\infty,+\infty]$. This coordinate (t,r) is called static coordinate. Substituting $\tan\theta = \sinh\rho$ in eq.(2.12), the metric takes form

$$ds^2 = \frac{L^2}{\cos^2\theta}\left[-d\tilde{t}^2 + d\theta^2\right] \tag{2.14}$$

where $\theta \in \left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$ and the spatial boundary at $\theta = \pm\frac{\pi}{2}$. This coordinate $(\tilde{t},\theta)$ is called conformal coordinate for AdS_2. Now we define another new coordinate system $X = Lrt$, $Y = \frac{Lr}{2}\left(-t^2 + \frac{1}{r^2} - 1\right)$, $Z = \frac{Lr}{2}\left(-t^2 + \frac{1}{r^2} + 1\right)$ and substitute in eq.(2.12), we find

$$\frac{ds^2}{L^2} = -r^2dt^2 + \frac{dr^2}{r^2} \tag{2.15}$$

where $t \in [-\infty,+\infty]$ and $r \in [0,\infty]$. Now we substitute $u = \frac{1}{r}$ in above equation, we get

$$ds^2 = \frac{L^2}{u^2}\left[-dt^2 + du^2\right] \ . \tag{2.16}$$

This coordinate (t,u) is called Poincaré coordinate for AdS_2. From above structure one can get H^2 space from AdS_2 spacetime by substituting $X_E = iX$(Euclidean).

Similarly one can get S^2 space from dS_2 spacetime by substituting $Z_E = iZ$.
$\boxed{AdS_{d+2}}$: We now generalize this AdS_2 to AdS_{d+2} spacetime. This spacetime must have $SO(2, d+1)$ invariance and is defined by

$$-X_0^2 - X_{d+2}^2 + X_1^2 + ... + X_{d+1}^2 = -L^2 \quad (2.17)$$
$$ds^2 = -dX_0^2 - dX_{d+2}^2 + dX_1^2 + ... + dX_{d+1}^2 \quad (2.18)$$

where L is the radius of AdS_{d+2} spacetime. In eq.(2.18), we substitute

$$X_0 = L\cosh\rho\cos\tilde{t},\ \ X_{d+2} = L\cosh\rho\sin\tilde{t},\ \ X_i = L\sinh\rho\ \hat{x}_i$$

where $\hat{x}_i(i = 1, 2, ...d+1)$ is the coordinate system of S^d. We find the metric in global coordinate system

$$ds^2 = L^2\left[-\cosh^2\rho\ d\tilde{t} + d\rho^2 + \sinh^2\rho\ d\Omega_d^2\right]\ . \quad (2.19)$$

Using similar procedure, we find the metric in different coordinate systems. In static coordinate system, the metric reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_d^2\ \ ;\quad f(r) = 1 + \frac{r^2}{L^2} \quad (2.20)$$

In conformal coordinate, it reads

$$ds^2 = \frac{L^2}{\cos^2\theta}\left[-d\tilde{t}^2 + d\theta^2 + \sin\theta^2 d\Omega_d^2\right] \quad (2.21)$$

Substituting

$$X_0 = \frac{Lr}{2}\left(x_i^2 - t^2 + \frac{1}{r^2} + 1\right), X_i = Lrx_i, X_{d+1} = \frac{Lr}{2}\left(x_i^2 - t^2 + \frac{1}{r^2} - 1\right), X_{d+2} = Lrt$$

in eq.(2.18), we find the metric

$$\frac{ds^2}{L^2} = -r^2\left(dt^2 + d\vec{x}_d^2\right) + \frac{dr^2}{r^2} \quad (2.22)$$

Using $z = \frac{1}{r}$, this metric reads in Poincarè coordinate

$$ds^2 = \frac{L^2}{u^2}\left[-dt^2 + du^2 + d\vec{x}_d^2\right]\ . \quad (2.23)$$

2.1.3 Some properties of AdS spacetime

This spacetime is maximally symmetric spacetime which has maximum number of symmetry generators [86]. There are mainly three spacetimes (Minkowski, de Sitter and Anti-de Sitter spacetime) which are maximally symmetric spacetime. The AdS_{d+2} have $SO(2, d+1)$ symmetry. The number of generators for Lorentz group and translation are $\frac{(d+1)(d+2)}{2}$ and $(d+2)$ respectively. So total number of symmetry generators is $\frac{(d+2)(d+3)}{2}$. The consequence of this symmetric spacetime is

$$R_{\alpha\beta\gamma\sigma} = \pm\frac{1}{L^2}\left(g_{\alpha\gamma}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\gamma}\right) \quad (2.24)$$

where $+$ and $-$ signs represent for positive curvature and negative curvature respectively. From the above expression, we find by contracting $g^{\alpha\gamma}$

$$R_{\beta\sigma} = \pm\frac{d+1}{L^2} g_{\beta\sigma} \ . \tag{2.25}$$

This type of spacetime are called Einstein spacetime ($R_{ab} \propto g_{ab}$) which satisfies Einstein field equation in vacuum

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0 \tag{2.26}$$

where Λ is the cosmological constant. By contracting $g^{\beta\sigma}$ in eq.(2.25), we find the Ricci scalar

$$R = \pm\frac{(d+1)(d+2)}{L^2} \ . \tag{2.27}$$

Substituting eq.(2.25,2.27) in eq.(2.26) and simplifying, we finally get

$$\begin{aligned} \pm\frac{d+1}{L^2} g_{\beta\sigma} \mp \frac{1}{2}\frac{d+1}{L^2} g_{\beta\sigma} + \Lambda g_{\beta\sigma} &= 0 \\ \Rightarrow \Lambda &= \pm\frac{d(d+1)}{2L^2} \ . \end{aligned} \tag{2.28}$$

So the maximal symmetric nature of spacetime fix the value of cosmological constant. For AdS_{d+2} spacetime this cosmological constant takes value $\boxed{\Lambda = -\frac{d(d+1)}{2L^2}}$.

Particle motion in AdS_2 spacetime

$\boxed{\textit{Photon motion}}$ Setting $L = 1$, we consider the static coordinate system for AdS_2 spacetime

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} \quad ; \quad f(r) = 1 + r^2 \tag{2.29}$$

The motion of photon have a different behaviour in AdS spacetime. For photon motion, we know that $ds^2 = 0$ which implies from eq.(2.29)

$$dt = \frac{dr}{f(r)} \Rightarrow t = \int_0^\infty \frac{dr}{1+r^2} = \frac{\pi}{2} \ . \tag{2.30}$$

The above expression shows that photon reaches the AdS boundary ($r = \infty$) in finite coordinate time.
$\boxed{\textit{Partcle motion}}$ Particle motion in AdS spacetime is also different than motion in Minkowski spacetime. We know that $p^\mu p_\mu = -m^2$ which implies for one unit mass

particle

$$
\begin{aligned}
p^0 p_0 + p^1 p_1 &= -1 \\
g^{tt} E^2 + g_{rr} \left(\frac{dr}{dt} \right)^2 &= -1 \\
\frac{1}{f(r)} \left(\frac{dr}{dt} \right)^2 &= -1 + \frac{E^2}{f(r)} \\
\frac{dr}{dt} &= \sqrt{E^2 - f(r)} \\
t &= \int_0^\infty \frac{dr}{\sqrt{E^2 - 1 - r^2}} \; . \qquad (2.31)
\end{aligned}
$$

This is the divergent integral which implies that particle in AdS spacetime can't reach the AdS boundary ($r = \infty$). However, the above expression shows that $\frac{dr}{dt}$ can take value positive and negative for $E > 1$. So there should a turning point (r_*) at which $\frac{dr}{dt} = 0$. This leads us $r_* = \sqrt{E^2 - 1}$. The particle goes to the turning point (r_*) in a finite time

$$
\begin{aligned}
t &= \int_0^{r_*} \frac{dr}{\sqrt{E^2 - 1 - r^2}} = \arcsin \left(\frac{r_*}{\sqrt{E^2 - 1}} \right) \\
&= \frac{\pi}{2} \qquad (2.32)
\end{aligned}
$$

In summary, we conclude that particle never reach the AdS boundary and it reflects back at a turning point in AdS spacetime whereas photon reaches AdS boundary in a finite coordinate time.

2.1.4 Emergence of AdS spacetime from scaling symmetry

We can write down a general metric in 5−dimensional spacetime which has Poincarè invariance in 4−dimensional spacetime [87]

$$
ds^2 = \Omega^2(z) \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \right) \qquad (2.33)
$$

where z is the extra coordinate. We now rescale the coordinate system by scaling $t \to \lambda t, x_i \to \lambda x_i$ and $z \to \lambda z$ which leads to

$$
ds^2 \to ds^2 = \lambda^2 \Omega^2(\lambda z) \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \right) \; . \qquad (2.34)
$$

The scale invariance implies

$$
\Omega^2(z) = \lambda^2 \Omega^2(\lambda z) \Rightarrow \Omega^2(z) = \frac{L^2}{z^2} \qquad (2.35)
$$

Substituting this Ω value in eq.(2.33), we get

$$
ds^2 = \frac{L^2}{z^2} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \right) \qquad (2.36)
$$

which is the AdS_5 spacetime in Poincarè coordinate system. Therefore, the scale invariance leads to AdS geometry. The above metric has $ISO(3,1)$ invariance on (t, x_i) which correspondence to the Poincarè invariance of dual gauge theory in 4−dimensional spacetime. This (t, x_i) coordinate is interpreted as spacetime coordinate of gauge theory. The extra coordinate z is interpreted as energy scale of dual gauge theory. In AdS/CFT, the coordinate time t is interpreted as the gauge theory time which implies that Hawking temperature in gravity theory is the temperature of the gauge theory [88]. We now would like to discuss about symmetry structure of any CFT in next section.

2.2 Symmetry of any CFT in d-dimensions

Let us start with the transformation in which under $x^\mu \to x'^\mu$, the line element transforms in following way [86]

$$dx'^\mu dx'_\mu = \Omega^2(x) dx^\mu dx_\mu \ . \tag{2.37}$$

For $\Omega(x) = \lambda$, this transformation is called scale transformation.
For $\Omega(x) = 1$, this transformation is called Lorentz transformation.
We assume infinitesimal transformation of $\Omega(x) = 1 + \frac{\omega(x)}{2}$ under $x'^\mu = x^\mu + v^\mu(x)$. So we can write from eq.(2.37)

$$dx'^\mu dx'_\mu = dx^\mu dx_\mu + \omega(x) dx^\mu dx_\mu + \mathcal{O}(\omega^2) \ . \tag{2.38}$$

If we put $x'^\mu = x^\mu + v^\mu(x)$ we find

$$dx'^\mu dx'_\mu = dx^\mu dx_\mu + (\partial_\mu v_\nu + \partial_\nu v_\mu) dx^\mu dx^\nu + \mathcal{O}(..) \ . \tag{2.39}$$

Comparing eq.(2.38) and eq.(2.39) upto linear order, we get

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x) \eta_{\mu\nu} \ . \tag{2.40}$$

Multiply above equation with $\eta^{\mu\nu}$, we find for d-dimension spacetime

$$\omega(x) = \frac{2}{d}\ \partial^\mu v_\mu \ . \tag{2.41}$$

Substituting eg.(2.41) in eq.(2.40), we find

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\ \partial^\alpha v_\alpha \eta_{\mu\nu} \ . \tag{2.42}$$

Multiply with ∂_ρ, the above equation becomes

$$\begin{aligned}
2\partial_\rho \partial_\mu \partial_\nu &= \frac{2}{d} \partial_\rho \partial^\alpha v_\alpha \eta_{\mu\nu} \qquad (\times \eta^{\rho\mu}) \qquad (2.43) \\
\Rightarrow 2\partial^2 v_\nu &= \frac{2}{d} \partial_\nu \partial^\alpha v_\alpha \qquad (\times \partial^\nu) \\
2\left(1 - \frac{1}{d}\right) \partial^2 \partial^\nu v_\nu &= 0 \\
\Rightarrow (d-1)\partial^2 \left[\frac{2}{d} \partial^\alpha v_\alpha\right] &= 0 \\
\Rightarrow \partial^2 \omega(x) &= 0 \qquad (2.44)
\end{aligned}$$

where $\omega(x) = \frac{2}{d}\partial^\alpha v_\alpha$. The general solution of the above equation is

$$\begin{aligned} \omega(x) &= A + B_\mu x^\mu \\ \Rightarrow \partial^\alpha v_\alpha &= \frac{d}{2}(A + B_\mu x^\mu) \end{aligned} \tag{2.45}$$

Substituting in eq.(2.43), we find

$$\partial_\rho \partial_\mu \partial_\nu = const. \tag{2.46}$$

The general solution of the above equation is

$$v^\mu = a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\rho\sigma} x^\rho x^\sigma \ . \tag{2.47}$$

where we later identify a^μ for translation, $b^\mu{}_\nu$ for Lorentz transformation and scaling transformation ($b^\lambda{}_\lambda$) and $c^\mu{}_{\rho\sigma}$ for special conformal transformation.
From eq.(2.40) one can show that $b_{\mu\nu}$ is anti-symmetric with the relation

$$b_{\mu\nu} + b_{\nu\mu} = \eta_{\mu\nu} \frac{2}{d} b^\lambda{}_\lambda \ . \tag{2.48}$$

So we can write $b_{\mu\nu} = \alpha\eta_{\mu\nu} + d_{\mu\nu}$ where α is for scaling symmetry and $d_{\mu\nu}$ is for Lorentz transformation. So total number of generators ($a^\mu \to d, b^\lambda{}_\lambda \to 1, b_{\mu\nu} = -b_{\nu\mu} \to \frac{d(d-1)}{2}$, special conformal transformation $\to d$) is $\frac{(d+1)(d+2)}{2}$ which is same as the number generators of $SO(2,d)$. Therefore one can conclude that the number of generators of CFT_d and AdS_{d+1} are same.

2.3 Pillar of AdS/CFT correspondence

To understand this duality in heuristic way, we start with D-branes physics in string theory. D-branes have masses which are proportional to $\frac{1}{g_s}$ [89]. At weak coupling string limit, they represents heavy objects. A p-dimensional brane is called Dp-brane. The low energy limit of a single D-brane theory is a $U(1)$ gauge theory. The stack of N D-brane is represented by $SU(N)$ gauge theory which is called SYM theory. Since D-branes are heavy object in limit of $g_s \to 0$, they curve spacetime around them which leads to classical supergravity theory. This is another picture of D-brane in which geometric description emerges. This duality means that there exists a precise map between fields on string theory (gravity side) and gauge invariant operators on the field theory side. The classical supergravity solution reads for stack of N D3-branes

$$ds^2 = H^{-1/2}(r)\left[-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + H^{1/2}(r)\left[dr^2 + r^2 d\Omega_5^2\right] \tag{2.49}$$

where $H(r) = \left(1 + \frac{L^4}{r^4}\right)$. Since we are working in low energy limit, we take $L >> r$ which implies $H(r) \approx \frac{L^4}{r^4}$. The metric becomes

$$ds^2 = \frac{r^2}{L^4}\left[-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{2.50}$$

which is precisely $AdS_5 \times S^5$. The symmetry group of this spacetime is $SO(2,4) \times SO(6)$. The radius of the five sphere is same as the radius (L) of AdS spacetime. It is also shown that [7]

$$\frac{L^4}{\alpha'^2} = 2g_{YM}^2 N \tag{2.51}$$

where α' is string scale. $SU(N)$ gauge theory is doable perturbatively when $g_{YM}^2 N << 1$. Using eq.(2.51), other limit ($g_{YM}^2 N >> 1$) of gauge theory can be analysed by constructing dual gravity theory which is reliable in $L >> \sqrt{\alpha'}$ limit. To construct dual gravity theory, we need a precise AdS/CFT dictionary. This dictionary will be discussed in this chapter by using GKPW formalism [5]-[6].

2.3.1 Brief discussion on $\mathcal{N} = 4$ Super Yang-Mills theory

$\mathcal{N} = 4$ SYM theory in four spacetime dimension is conformally invariant $SO(2,4)$, which contains gauge fields, 6 scalar and 4 fermion. There is $SU(4)$ R-symmetry which rotates the scalar and the fermions among themselves. The symmetry group of this theory is $SO(2,4) \times SU(4)$. We start with discussion on basic of Yang-Mills theory [8].
Yang-Mills theory is the non-abelian generalization of Maxwell's theory. The action for Yang-Mills (YM) theory is

$$S_{YM} = -\frac{1}{2g^2} \int \text{Tr.}(F_{\mu\nu} F^{\mu\nu}) \tag{2.52}$$

where g is the Yang-Mills coupling term, $F_{\mu\nu} = [D_\mu, D_\nu] = A_{\nu,\mu} - A_{\mu,\nu} + ig[A_\mu, A_\nu]$ and $A_\mu = A_\mu^a t^a$ (t^a is generators of $SU(N)$ group). This generators satisfies $\text{Tr.}(t^a t^b) = \frac{1}{2}\delta^{ab}$ relation. For $SU(N)$ gauge theory corresponds to $N \times N$ matrix representation of generators t^a where $a = 1, 2, ...(N^2 - 1)$. We now impose supersymmetry in this theory. This symmetry tells us about duality between bosons and fermions. There exists supercharge Q_s which generates fermion(F)-boson(B) transformation. This symmetry removes lot of divergence in various calculations which occurs due to bosonic field in that theory. The mathematical statement about supersymmetry is

$$Q_s B \to F \qquad \& \qquad Q_s F \to B$$

with the property $[Q_s, H] = 0$ where H is the Hamiltonian of the theory.
Consider $\mathcal{N} = 1$ super Yang-Mills (SYM) theory where $\mathcal{N}$ denotes the number of super charge Q_s which generate fermion/boson by acting on boson/fermion. This theory contains two fields : one boson field A_μ and one fermion field ψ_α. Here supercharge operator is δ which gives the following transformation

$$\delta A_\mu \to \psi_\alpha \qquad \& \qquad \delta\psi_\alpha \to F_{\mu\nu}$$

where number of degree of freedom for both fields are 2.
For $\mathcal{N} = 2$ SYM theory the above relation is following :

$$\begin{aligned} \delta_1 A_\mu &\to \psi_{1\alpha} \qquad , \qquad \delta_1\psi_{2\alpha} \to \phi \\ \delta_2 A_\mu &\to \psi_{2\alpha} \qquad , \qquad \delta_2\psi_{1\alpha} \to \phi \end{aligned}$$

Degree of freedom for boson fields is $4(A_\mu = 2, \phi = 2)$ and d.o.f of fermion fields is also 4 (each field have 2 d.o.f.). One can now generalize it this theory for large number of $\mathcal{N}$.
For $\mathcal{N} = 4$ SYM theory, the number of degree of freedom for fermion fields ($\phi^{\pm}_{i\alpha}(i = 1, .., 4.)$) and gauge fields ($A_\mu, \phi_a(a = 1, .., 6.)$) are 8. The four supercharge $Q_s(s = 1, .., 4)$ can combine different combinations which imply that there exists a symmetry which rotates four supercharges without affecting $\mathcal{N} = 4$ supersymmetry. It implies that this theory have $SU(4)$ symmetry. There are 6 real scalar ϕ_a in this theory which implies that this scalar have adjoint representation in this theory. The adjoint representation is given by anti-symmetric field $B_{ij} = -B_{ji}$. Here fermion are fundamental representation. One can show $SU(4) \simeq SO(6)$. The action for this theory is [8]

$$\begin{aligned} S_{SYM} &= -\frac{1}{g^2}\int d^4x \text{ Tr.}\left\{F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{4}\bar{\psi}_{i\alpha}\not{D}\psi_{i\alpha} + \sum_{a=1}^{6}(D_\mu\phi)^\dagger(D_\mu\phi)\right. \\ &+ \left.\sum_{a,i}\bar{\psi}_{i\alpha}B_{ij}\psi_{i\alpha} + \sum_{a,b}([\phi_a, \phi_b])^2\right\} + \frac{\theta}{16\pi^2}\int d^4x \text{Tr.}(F_{\mu\nu}F_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{2.53}$$

where $B_{ij} = \phi_a\tau^a_{ij}$ (τ^a is 4×4 anti-symmetry matrix). This theory does not allow any mass term. So it is classically conformal invariant theory. It is also conformal invariant in quantum version. The symmetry of this theory is $SO(2, 4)$. It also have $SU(4)$ symmetry, which is often called R-symmetry. Finally we conclude that $\mathcal{N} = 4$ SYM theory has $SO(2, 4) \times SO(6)$ symmetry since $SU(4) \simeq SO(6)$. In final remark, we say that the symmetry group of this theory is equivalent to the symmetry group of low energy limit of classical supergravity theory (2.50).

2.3.2 Brief discussion on String theory and D-brane physics

Action for free particle which is parameterized by τ reads

$$S = -m\int ds = -m\int d\tau\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{2.54}$$

where $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. There are two problem in this form of action. This form of action is difficult to quantize and this does not work in massless limit. To resolve these issues we have to introduce an auxiliary field $e(x)$ which is called 'tetrad'. The action for free particle now reads

$$S = -\frac{1}{2}\int d\tau\left[e^{-1}(x)\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - m^2e(x)\right] . \tag{2.55}$$

We can now generalize this action for string whose dynamics is described two parameters τ and σ. Therefore, the action for string dynamics [80]-[83]

$$S = -T\int d^2\zeta\sqrt{-det.\left(\eta_{\mu\nu}\frac{\partial x^\mu}{\partial\zeta^\alpha}\frac{\partial x^\nu}{\partial\zeta^\beta}\right)} \tag{2.56}$$

where $T=\frac{1}{2\pi l_s}$ is the tension of string and $\zeta^0=\tau$ and $\zeta^1=\sigma$ and l_s is the string length. In low energy limit, $l_s\to 0$ and $T\to\infty$. In this form it is difficult to quantize. The different form of the above action reads [80]

$$S=-\frac{T}{2}\int d^2\zeta\sqrt{-h}h^{\alpha\beta}\partial_\alpha\chi^\mu\partial_\beta\chi^\nu\eta_{\mu\nu}\ . \tag{2.57}$$

This action is called super string theory action because it has conformal symmetry. When we want to quantize it, this theory losses it's conformal symmetry and demand for 10 dimensional spacetime. This theory contains massless spin-0 (scalar-dilaton), spin-1 (antisymmetry 2-form field which is called Ramond-Ramond field) and spin-2 (graviton) particles. Now we generalize the above action in curved space which becomes [8]

$$S=-\frac{T}{2}\int d^2\zeta\left[\sqrt{-h}h^{\alpha\beta}\partial_\alpha\chi^\mu\partial_\beta\chi^\nu g_{\mu\nu}+\partial_\alpha\chi^\mu\partial_\beta\chi^\nu\epsilon^{\alpha\beta}B_{\mu\nu}+\alpha'\sqrt{-h}\phi R^{(2)}\right]\ . \tag{2.58}$$

This leads us to equation of motions of 2 dimensional string and some constraints. The equivalent action upto $\mathcal{O}(\alpha')$ in 10 dimensional background field is

$$S=\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g}e^{-2\phi}\left[R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{2\times 3!}H_{\mu\nu\rho}H^{\mu\nu\rho}+\mathcal{O}(\alpha')\right] \tag{2.59}$$

which is known as the super-gravity action. In the full form, super-gravity action contains $g_{\mu\nu},\phi,B_{\mu\nu},\chi,A_{\mu\nu},A^\dagger_{\mu\nu\rho\sigma}$ etc. The full form of the super-gravity action is [8]

$$\begin{aligned}S &= \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}\left[R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{2\times 3!}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi\right.\\ &\left.-\frac{1}{2\times 3!}(F_{\mu\nu\rho}+\chi H_{\mu\nu\rho})(F^{\mu\nu\rho}+\chi H^{\mu\nu\rho})-\frac{1}{2\times 5!}F_{\mu\nu\rho\sigma\tau}F^{\mu\nu\rho\sigma\tau}\right)\\ &+\frac{1}{32\pi G_{10}}\int A_{[4]}\wedge F_{[3]}\wedge H_{[3]}\end{aligned} \tag{2.60}$$

where 5-form field F is self-dual. On the basis of chirality this super-gravity action takes type IIB super-gravity action, where fields has same chirality. The simplest form of the above action is

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left[R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2\times 5!}F^2_{[5]}\right] \tag{2.61}$$

where $\kappa^2=8\pi G_{10}$ and we neglect other fields in (2.60). One of the solution of this action is called $D3$ brane. For $D3$ brane, we consider $B_{\mu\nu}=0,\chi=0,A_{\mu\nu}=0$ and $H_{\mu\nu\rho}=0$. This action is in Einstein frame which is related to string frame by $g^{string}_{\mu\nu}=e^{\phi/2}g^{Einstein}_{\mu\nu}$. The above action leads to the following equations

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2\times 4!}F_{\mu\alpha_1\alpha_2\alpha_3\alpha_4}F_\nu{}^{\alpha_1\alpha_2\alpha_3\alpha_4} &= 0 \qquad (2.62)\\ \partial_\mu\left[\sqrt{-g}F^{\mu\alpha_1\alpha_2\alpha_3\alpha_4}\right] &= 0 \qquad (2.63)\\ \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\partial^\mu\phi\right] &= 0\ . \qquad (2.64)\end{aligned}$$

The $D3$-brane solution reads

$$ds^2 = H^{-1/2}(r)\left[-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + H^{1/2}(r)\left[dr^2 + r^2 d\Omega_5^2\right] \tag{2.65}$$

where $H(r) = \left(1 + \frac{L^4}{r^4}\right)$ and L is the AdS radius. The string coupling g_s is related with field by $g_s = \langle e^{\phi} \rangle$. The $'r'$ is the transverse direction which is defined as $r = \sqrt{(x^4)^2 + ... + (x^9)^2}$ and $r = 0$ is the horizon. This spcetime structure is asymptotically flat and $\nabla_6^2 H(r) = 0$. In the low energy limit $(r << L)$, $H(r) \approx \frac{L^4}{r^4}$ and the above metric becomes

$$ds^2 = \frac{r^2}{L^4}\left[-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{2.66}$$

which is $AdS_5 \times S^5$ geometry. This is the $AdS_5 \times S^5$ spacetime with $SO(2,4) \times SO(6)$ symmetry group.

So the low energy limit of $D3$ brane solution gives AdS spacetime.

Using the calculation of mass of D3-brane, one can show the relation between the AdS radius L and the string length α' [8]

$$\boxed{L^4 = 4\pi g_s \alpha'^2 N} \tag{2.67}$$

where g_s is the string coupling.

Relation between string coupling & Yang-Mills coupling

We know the string action in a sheet which can generalized for Dp-brane as [90],[91]

$$S_{Dp} = -T_{Dp}\int d^{p+1}\zeta \sqrt{-det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} \tag{2.68}$$

where $g_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \zeta^{\alpha}}\frac{\partial x^{\nu}}{\partial \zeta^{\beta}}\eta_{\mu\nu}$ and $\mu, \nu = 0, 1, .., q$ and $\alpha, \beta = 0, 1, .., p$ and $F_{\alpha\beta}$ is field tensor. This action is known as Dirac-Born-Infeld (DBI) action in literature. We know that $det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}) = det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})^T = det(g_{\alpha\beta} - 2\pi\alpha' F_{\alpha\beta})$. Using this fact we simplify

$$\begin{aligned}
\sqrt{-det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} &= \left[(-det\ g_{\alpha\beta})^2 (det(1 + 2\pi\alpha' g^{-1} F_{\alpha\beta}))^2\right]^{1/4} \\
&= \sqrt{-g}\left[det(1 - 4\pi^2\alpha'^2 g^{-2} F^2)\right]^{1/4} \\
&= \sqrt{-g}\left[e^{\frac{1}{4}\mathrm{Tr}.log(1 - 4\pi^2\alpha'^2 g^{-2} F^2)}\right] \\
&= \sqrt{-g}\left[1 - \frac{1}{4}4\pi^2\alpha'^2 tr.(g^{-1}F)^2 - ...\right] \\
&= \sqrt{-g}\left[1 + \frac{1}{4}4\pi^2\alpha'^2 F_{\alpha\beta}F^{\alpha\beta} + ...\right] .
\end{aligned} \tag{2.69}$$

Substitute in the action of D3-brane, we obtain

$$S_{D3} = -\frac{1}{(2\pi)^3 g_s \alpha'^2}\int d^4\zeta \sqrt{-g}\left[1 + \pi^2\alpha'^2 F_{\alpha\beta}F^{\alpha\beta} + ...\right] \tag{2.70}$$

The Yang-Mills action turns out in expansion

$$S_{YM} = -\frac{1}{-g_{YM}^2}\int d^4x\sqrt{-g}\left[\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + ...\right] \tag{2.71}$$

Compare equations (2.70) and (2.71), we find

$$\boxed{g_{YM}^2 = 4\pi g_s}\ . \tag{2.72}$$

The 't Hooft coupling is defined as $\lambda = g_{YM}^2 N$. For $\lambda >> 1$, string theory turns into purely gauge theory.

2.4 AdS/CFT correspondence

Maldacena's Conjecture : $\mathcal{N} = 4$ SYM with $SU(N)$ gauge symmetry is dual to Type IIB string theory on $AdS_5 \times S^5$.

The link between gauge theory and string theory was first observed in perturbative expansion of both theories. G. t' Hooft [92] had taken the color quantum number N as parameter and promoted A_μ to $N \times N$ matrix to solve gauge theory in all range of coupling constant g_{YM}. The perturbative expansion of $SU(N)$ gauge theory can be now organized in terms of $\frac{1}{N}$ in which the computations of theory simplify in the large N limit. The large N limit ($N \to \infty$) implies to $g_{YM} \to 0$ in which $\lambda = g_{YM}^2 N$ is kept constant. The parameter λ is called the t' Hooft parameter. This $\frac{1}{N}$ expansion in gauge theory links with the perturbative expansion with the string coupling constant g_s in string theory [93]. Using perturbative expansion, it was shown that large N gauge theory is equivalence to string theory with some identification. The glueball (single trace operator) in gauge theory and topology of Feynmann diagram are equivalent to external string state and topology of string worldsheet in string theory respectively [8]. In the limit of $g_s \to 0$, string theory corresponds to classical supergravity theory since the string spectrum always contains the graviton. Dp-brane is dynamical object in $p + 1$-dimensional hyperplane which emerges from string theory with Dirichlet boundary condition [84]. The low energy limit of a single D-brane theory is a $U(1)$ gauge theory. The stack of N D-brane is represented by $SU(N)$ gauge theory which is called Super Yang-Mills (SYM) theory. This is the one version of D-brane physics in low energy limit.

There is another version of D-brane physics which is geometry description. In previous sections, we have shown how the AdS geometry emerges from string theory. Dp-brane has a mass inversely proportional to string coupling g_s. In low energy limit ($g_s \to 0$), Dp-brane acts like heavy object which leads to dynamical spacetime (curve spacetime). This curve spacetime is nothing but $AdS_5 \times S^5$ geometry in which string dynamics lives.

These two pictures came out from same physics, in other words, they are equivalent. Two descriptions of D3-brane establish the Maldacena's conjecture. Lets consider D-branes in flat 10 dimensional Minkowski spacetime where open string live with

Dirichlet boundary condition. The low energy limit define as $\alpha' E^2 \to 0$ where α' and E are string length and energy scale of string theory. In previous section, we have showed in low energy limit that the coupling of gauge theory (g_{YM}) is related with string coupling g_s as $g_{YM} = 4\pi g_s$. In this limit, we decouple open string (gauge theory) and closed string (gravitons) as

$$\mathcal{N} = 4 \; SYM \; theory \;\; in \; d = 4 \; spacetime \; dimension \;+\; free \; gravitons \tag{2.73}$$

In second version of the D3-brane, it is described by the geometry in metric (2.65). The energy of this string picture is defined with respect to time t at $r = \infty$. In the low energy limit $\alpha' E^2 \to 0$, all massive closed string decouples and D3-brane acts as

$$TypeIIB \; string \; theory \; in \; AdS_5 \times S^5 \; (F_5 \; flux) + \; free \; gravitons \; at \; r = \infty \tag{2.74}$$

Comparing this two picture (2.73) and (2.74), we can conclude that $\mathcal{N} = 4$ SYM in $d = 4$ is equivalence to type IIB string in AdS_5 with F_5 flux on S^5. This is the heuristic way to derive AdS/CFT correspondence [8].

2.5 AdS/CFT Dictionary

GKPW Prescription [5],[6] : The partition function of a gauge theory with scale invariance is equivalent to the partition function of string theory on AdS_5 spacetime which describes classical gravity theory in low energy limit. The mathematical statement is

$$Z_{QFT} = Z_{AdS_5} \; . \tag{2.75}$$

The partition function of gauge theory is

$$Z_{QFT}[\phi_0] = \int \mathcal{D}A e^{i\{S_{QFT}[A] + \phi_0 \hat{\mathcal{O}}(A)\}} \tag{2.76}$$

where ϕ_0 is the external source field and $\hat{\mathcal{O}}$ is a particular operator in gauge theory. For strongly coupled QFT, it is difficult to compute Z_{QFT}. Using GKPW formula one can compute Z_{QFT} for strongly coupled QFT in 4 dimensional spacetime by studying dual gravity theory in AdS_5. From gravity side, we can compute Z_{AdS_5} by saddle point approximation which reads

$$Z_{AdS_5} \approx e^{i\bar{S}_{gr}[\phi \to \phi_0]} \tag{2.77}$$

where $\bar{S}_{gr}$ is the on-shell action for classical gravity and ϕ_0 is bulk field measured at boundary of the spacetime. Since the bulk field $\phi(z, x)$ satisfies equation of motion, the on-shell action reduce to surface term on AdS boundary. This surface term is the generating functional of gauge theory which lives on the boundary of AdS_5. Therefore we can write

$$Z_{QFT} \approx e^{i\bar{S}_{gr}[\phi \to \phi_0]} \tag{2.78}$$

where ϕ is dynamical bulk field in 5-dimensional spacetime and ϕ_0 is the external source field of the boundary field theory in 4-dimensional spacetime. This two field is related by

$$\phi_0 = \lim_{z \to 0} \phi(z) \tag{2.79}$$

where $z = 0$ represent the boundary of AdS_5 spacetime. This implies that external source of the 4-dimensional field theory have 5-dimensional origin. The one-point function in field theory can be calculated by GKPW formula. In the presence of external source, one-point function reads

$$\langle \hat{\mathcal{O}} \rangle_s = \frac{\delta \bar{S}_{gr}[\phi_0]}{\delta \phi_0} \tag{2.80}$$

where $\hat{\mathcal{O}}$ is a particular operator or the response to the external field ϕ_0. Without external source, this reads as

$$\langle \hat{\mathcal{O}} \rangle = \left. \frac{\delta \bar{S}_{gr}[\phi_0]}{\delta \phi_0} \right|_{\phi_0 = 0} . \tag{2.81}$$

From linear response theory we will see that the perturbation on the bulk field corresponds to response function which is the operator in the boundary field theory.

Use of GKPW formula: The perturbation theory fails to describe strongly coupled system. To describe strongly coupled systems, one can construct dual theory which is weakly coupled and is analyzed by perturbation theory. It is not an easy task always to find the map $S_{QFT} \to S_{gravity}$. The dual theory of $\mathcal{N} = 4$ SYM is type IIB string theory on $AdS_5 \times S^5$. Although there are some particular example, the general proof of this map does not exist till date. To define QFT on boundary, we construct a classical gravity model which is dual to large N QFT or strongly coupled QFT. This is the easiest way to construct dual theory.

The classical gravity action in AdS_5 is

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\, (R - 2\Lambda) \tag{2.82}$$

This solution of this action is

$$ds^2 = \frac{1}{z^2} \left(-f(z) dt^2 + d\vec{x}^2 \right) + \frac{dz^2}{f(z) z^2} \quad ; \quad f(z) = 1 - z^4 \tag{2.83}$$

where $d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2$, $z = 1$ is the horizon and $z = 0$ is the boundary of the spacetime. In asymptotic limit ($z \to 0$), we find that $f(z) \approx 1$ and the metric becomes

$$ds^2 = \frac{1}{z^2} \left(-dt^2 + d\vec{x}^2 + dz^2 \right) . \tag{2.84}$$

For AdS/CFT dictionary, we start with some example of field theory which are given below.

1. The massless scalar field theory : The matter action in AdS_5 spacetime is given by [87]

$$S = -\frac{1}{2}\int d^5x\sqrt{-g}\left(\partial_\mu\phi\partial^\mu\phi\right) \ . \tag{2.85}$$

The asymptotic AdS_5 spacetime is given by eq.(2.84). For simplicity, we consider static homogeneous solution of the scalar field $\phi \equiv \phi(z)$. Using metric (2.84), we obtain from eq.(2.85)

$$\begin{aligned} S &= -\frac{1}{2}\int d^4x dz \frac{1}{z^5}\partial_z\phi\partial^z\phi \\ &= -\frac{1}{2}\int d^4x \int_0^1 dz \frac{(\partial_z\phi)^2}{z^3} \\ &= -\frac{1}{2}\int d^4x \int_0^1 dz \left[\partial_z\left(\frac{\phi\partial_z\phi}{z^3}\right) - \phi\partial_z\left(\frac{\partial_z\phi}{z^3}\right)\right] \end{aligned} \tag{2.86}$$

where first term is the total derivative term and second term is the equation of motion for the scalar field which implies

$$\partial_z\left(\frac{\partial_z\phi}{z^3}\right) = 0 \ . \tag{2.87}$$

Substituting eq.(2.87) in eq.(2.86) we find the on-shell action

$$\bar{S} = \frac{1}{2}\int d^4x \frac{\phi\partial_z\phi}{z^3}\bigg|_{z=0} \tag{2.88}$$

in which we use the regularity condition of scalar field $\phi(1) = 0$. The solution of the equation of motion (2.87) is

$$\phi(z) = c_1 + \frac{c_2}{2}z^4 \tag{2.89}$$

where c_1 and c_2 are integration constant. We know that

$$\begin{aligned} \phi_0 &= \lim_{z\to 0}\phi(z) \\ &= c_1 \ . \end{aligned} \tag{2.90}$$

Substitute $c_1 = \phi_0$ and $c_2 = \phi_0\phi_1$ in eq.(2.89), we find

$$\phi(z) = \phi_0(1 + \phi_1 z^4) \tag{2.91}$$

Putting eq.(2.91) in eq.(2.88), the on-shell action becomes

$$\bar{S}[\phi_0] = \int d^4x 2\phi_0^2\phi_1 \tag{2.92}$$

which is the generating functional of the boundary gauge theory. From GKPW formula the one-point function takes the form

$$\begin{aligned} \langle\hat{\mathcal{O}}\rangle_s &= \frac{\delta\bar{S}[\phi_0]}{\delta\phi_0} \\ &= 4\phi_0\phi_1 \ . \end{aligned} \tag{2.93}$$

Now the scalar field solution becomes

$$\phi(z) = \phi_0 + \frac{\langle\hat{\mathcal{O}}\rangle_s}{4} z^4 \tag{2.94}$$

where first term is the slow fall off field and second term represents the fast off field. From GKPW prescription, we know that ϕ_0 is the external source of the boundary gauge theory and $\langle\hat{\mathcal{O}}\rangle_s$ is the operator of the boundary theory associated with this field. From linear response theory, these operators $\langle\hat{\mathcal{O}}\rangle_s$ are the response function under external source ϕ_0. From this, we conclude that the fast fall off field and the slow fall off field represent the response and external source of the boundary gauge theory. Now one needs to guess the boundary operator corresponding to bulk field. The response functions should be conserved quantities of the theory such as $T^{\mu\nu}$, J^μ etc. Hence the boundary operators are related to $T^{\mu\nu}$, J^μ, charge density etc. The boundary operator $\langle\hat{\mathcal{O}}\rangle_s$ depends on the external field ϕ_0 which corresponds the bulk field obtained by promoting this field into 5-dimensional dynamical field. From the linear response theory, we know that the response

$$\langle\hat{\mathcal{O}}\rangle_s = -\int d^4x' G_R^{00}(x - x')\phi_0(x') \tag{2.95}$$

where G_R^{00} is the green function. Compare eq.(2.93) and eq.(2.95), we find

$$G_R^{00}(0) = -4\phi_1 \ . \tag{2.96}$$

The scaling dimension of this field is determined by the scaling symmetry of this theory. The scalar field transforms as

$$\phi \to a^{-\Delta}\phi$$

under the scaling symmetry of the spacetime

$$x^\mu \to a x^\mu \quad ; \quad z \to az \ .$$

This Δ is called scaling dimension of the field. From eq.(2.94), we see that under this above scaling symmetry

$$\phi_0 \to \phi_0 \quad : \quad \Delta = 0$$

and

$$\langle\hat{\mathcal{O}}\rangle_s \to a^{-4}\langle\hat{\mathcal{O}}\rangle_s \quad : \quad \Delta = 4.$$

Then the perturbed action for boundary field theory reads

$$\delta S_{QFT} = \frac{1}{2}\int d^4x \phi_0 \hat{\mathcal{O}} \tag{2.97}$$

which is scale invariant theory under this scaling symmetry. From eq.(2.92) and eq.(2.93), we find $\bar{S}[\phi_0] = \delta S_{QFT}$; in other words, we say that the perturbed boundary action is constructed from the on-shell bulk action.

2. The massive scalar field : The action for massive scalar field in AdS_5 is

$$S = -\frac{1}{2}\int d^5x\sqrt{-g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right) \ . \tag{2.98}$$

Using Euler-Lagrangian equation, we obtain the equation of motion for massive scalar field from eq.(2.99)

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) - m^2\phi = 0\ . \tag{2.99}$$

Using previous anstz of $\phi \equiv \phi(z)$ and considering asymptotic AdS_5 metric (2.84), we find equation of motion

$$z^2\partial_z^2\phi - 3z\partial_z\phi - m^2\phi = 0\ . \tag{2.100}$$

Using this equation of motion, we obtain the on-shell action similarly

$$\bar{S} = \frac{1}{2}\int d^4x \frac{\phi\partial_z\phi}{z^3}\bigg|_{z=0}\ . \tag{2.101}$$

To know the solution of the eq.(2.100), we consider $\phi \sim z^\Delta$. Substituting this in eq.(2.100), we find

$$\Delta^2 - 4\Delta - m^2 = 0\ . \tag{2.102}$$

The roots of this quadratic equation fix the value of Δ which is

$$\Delta_\pm = 2 \pm \sqrt{4+m^2}. \tag{2.103}$$

This implies that Δ_+ is always greater than Δ_- for any value of m^2 and $(\Delta_+ + \Delta_-) = 4$. So the asymptotic solution reads

$$\phi(z) = Az^{\Delta_-} + Bz^{\Delta_+}\ . \tag{2.104}$$

The value of Δ should be real which implies from eq.(2.103)

$$m^2 \geq -4\ . \tag{2.105}$$

This is called Breitenlohner-Freedman bound [94],[95] which determines the stability of the theory. The negative of the square of mass is allowed for the stable QFT in AdS_5. From eq.(2.104), we see that the first term is slow fall off term and the second term is fast fall off term. Frow GKPW we know that slow fall off term and fast fall off term are the source field and response field respectively. This in turn implies that $A = \phi_0$ and $B \sim c_\phi\langle\hat{\mathcal{O}}\rangle_s$ where c_ϕ is just a factor. The eq.(2.104) reads now

$$\phi(z) \sim \phi_0 z^{\Delta_-} + c_\phi\langle\hat{\mathcal{O}}\rangle_s z^{\Delta_+}\ . \tag{2.106}$$

Substituting eq.(2.106) in eq.(2.101), we obtain

$$\begin{aligned}
\bar{S} &= \frac{1}{2}\int d^4x \frac{1}{z^3}\left[\left(\phi_0 z^{\Delta_-} + c_\phi\langle\hat{\mathcal{O}}\rangle_s z^{\Delta_+}\right)\left(\Delta_-\phi_0 z^{\Delta_- -1} + c_\phi\Delta_+\langle\hat{\mathcal{O}}\rangle_s z^{\Delta_+ -1}\right)\right]\bigg|_{z=0} \\
&= \frac{1}{2}\int d^4x \left[\left\{(\Delta_+ + \Delta_-)c_\phi\phi_0\langle\hat{\mathcal{O}}\rangle_s z^{(\Delta_+ + \Delta_-)-4}\right\} + \left\{\phi_0^2\Delta_- z^{2\Delta_- -4} + \langle\hat{\mathcal{O}}\rangle_s^2\Delta_+ z^{2\Delta_+ -4}\right\}\right]\bigg|_{z=0} \\
&= \frac{1}{2}\int d^4x \left[4c_\phi\phi_0\langle\hat{\mathcal{O}}\rangle_s + 0\right]
\end{aligned} \tag{2.107}$$

where the second term is zero when $\Delta_\pm \geq 2$. This on-shell action represents the perturbed boundary field theory action which reads

$$\delta S_{QFT} \sim \int d^4x \phi_0 \langle \hat{\mathcal{O}} \rangle_s \ . \tag{2.108}$$

3. The Maxwell Field : Consider a boundary field theory with $U(1)$ current J^μ which is explicitly specified. When J^0 represents the charge density associated with the chemical potential μ, there should be a Maxwell's field A_0 in the bulk action which is dual to the boundary theory. The bulk action in AdS_5 reads

$$S = -\frac{1}{4}\int d^5x \sqrt{-g} F^{\mu\nu} F_{\mu\nu} \tag{2.109}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is field tensor. The equation of motion for gauge field is

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu\nu}\right) = 0 \ . \tag{2.110}$$

For simplicity, we consider $A_\mu = (A_0(z), 0, 0, 0, 0)$. Substituting this in the above equation, we get

$$\partial_z \left(\frac{\partial_z A_0}{z}\right) = 0 \ . \tag{2.111}$$

The solution of the above equation reads

$$A_0(z) = A_0^{(0)}(1 + A_0^{(1)} z^2) \tag{2.112}$$

where $A_0^{(0)}$ and $A_0^{(1)}$ are the integration constant. The first term and second term represent the slow fall off field and fast fall off field. The chemical potential of the boundary theory is defined as

$$\mu = \lim_{z\to 0} A_0(z) = A_0^{(0)} \ . \tag{2.113}$$

Using this ansatz the action (2.109) becomes

$$\begin{aligned} S &= -\frac{1}{4}\int d^4x dz \left(-\frac{(\partial_z A_0)^2}{z}\right) \\ &= \frac{1}{4}\int d^4x \int_0^1 dz \left[\partial_z\left(\frac{A_0 \partial_z A_0}{z}\right) - \partial_z\left(\frac{\partial_z A_0}{z}\right) A_0\right] \ . \end{aligned} \tag{2.114}$$

Using the equation of motion, we find the on-shell action which reads

$$\begin{aligned} \bar{S} &= \frac{1}{4}\int d^4x \int_0^1 dz \left[\partial_z\left(\frac{A_0 \partial_z A_0}{z}\right)\right] \\ \bar{S} &= \int d^4x \frac{A_0 \partial_z A_0}{4z}\bigg|_{z=0} \end{aligned} \tag{2.115}$$

Using the gauge field solution (2.112), the eq.(2.115) becomes

$$\bar{S} = -\frac{1}{2}\int d^4x A_0^{(0)2} A_0^{(1)} \ . \tag{2.116}$$

Table 2.1: *AdS/CFT* dictionary

Gauge theory (CFT_d)	Gravity theory (AdS_{d+1})
QFT Temperature T	Hawking's temperature T_H
Condensate ($\langle\mathcal{O}\rangle$)	Charged scalar matter field (ψ)
Global symmetries	Gauged symmetries
Current conservation	Gauge invariance
Conformal dimensions Δ	Mass m^2
Stress tensor $T^{\mu\nu}$	metric $g_{\mu\nu}$
Current J^μ - charge density ρ	Gauge field A_μ
QFT phase transitions	Black hole instabilites

The response function for the gauge field should be current which reads

$$\langle J^0\rangle_s = \frac{\delta\bar{S}}{\delta A_0^{(0)}} = -A_0^{(0)}A_0^{(1)} \ . \tag{2.117}$$

From GKPW prescription this response $\langle J^0\rangle_s$ is the charge density ρ of the boundary theory which implies $\rho = -A_0^{(0)}A_0^{(1)}$. The gauge field in terms of quantities of boundary theory reads now

$$A_0 = \mu - \rho z^2 \ . \tag{2.118}$$

For A_x one can calculate similarly the current $\langle J^0\rangle_s = c_A A_0^{(0)}A_0^{(1)}$ where c_A is just a factor. Similary, other quantities in gravity theory can be mapped to the quantities in gauge theory. Using this formalism, *AdS/CFT* dictonary has been established explictly in [5]-[6]. The map between quantities in gauge side and quantities in gravity side are given in Table 2.1.

2.6 Holography

The specific heat of any statistical system is positive and entropy is proportional to volume of the system. But for gravitational system, the specific heat is positive only for planar geometry and it's entropy is proportional to area. The planar geometry play a important role in studying non-gravitational system using gague/gravity duality. Using a dual gravitational theory one can describe a phenomena of a non-gravitational system. But the measure of entropy of two theories behave differently. So we need to consider one higher dimensional gravity theory to describe a phenomena of a non-gravitational system which lives on the boundary of higher dimensional spacetime. Then we can argue that degrees of freedom of gravitational theory in bulk and degrees of freedom of non-gravitational system on boundary are same because the degrees of freedom of gravitational theory is counted at boundary of that spacetime.

For non-gravitational system, a region is fully described by it volume. In hologram, three dimension information is encoded in two dimensional picture. The information of a volume is encoded in it's area for gravitational theory. That's why the name

"Holographic" came into the gravitational theory.

Consider isolated system of mass E and entropy S_0 in asymptotic flat spacetime with area A which encloses the system. Let M_A be the mass of black hole with horizon area A. This implies that [85]

$$E \leq M_A \tag{2.119}$$

in which $(M_A - E)$ amount of energy is added to the system, keeping A fixed. Now the black hole entropy is

$$S_{BH} \geq S_0 + S_1 \tag{2.120}$$

where S_1 is the entropy due to extra added energy. This implies that

$$S_0 \leq S_{BH} = \frac{A}{4\hbar G_N} \ . \tag{2.121}$$

This tells us that maximum entropy inside a region bounded by area A is given by

$$S_{max} = \frac{A}{4\hbar G_N} \ . \tag{2.122}$$

We treat black hole as a quantum statistical object in which entropy is defined by

$$S = -Tr.\rho \log \rho \ . \tag{2.123}$$

where ρ is density operators for system. For a system with N-dimensional Hilbert space, the density operator $\rho = \frac{1}{N}\mathbb{I}$ which leads us to the maximum entropy

$$S_{max} = \log N \ . \tag{2.124}$$

Compare with eq.(2.122) and eq.(2.124), we find

$$\log N = \frac{A}{4\hbar G_N} = \frac{A}{4l_p^2} \tag{2.125}$$

where l_p is Planck length.
Holographic Principle [2, 3] : In quantum gravity, a region of boundary area A can be fully described by $\frac{A}{4l_p^2}$ degrees of freedom that is one degree per Planck area.

Counting degrees of freedom : Consider a CFT$_d$ in a box of size L and put it in a lattice with lattice spacing ϵ [87]. The number of cells is $\left(\frac{L}{\epsilon}\right)^{d-1}$. If c is the number of degrees of freedom of each cell, the total number of degrees of freedom is

$$N_{dof}^{CFT} = \left(\frac{L}{\epsilon}\right)^{d-1} c \tag{2.126}$$

For any CFT, this c is identified as central charge of the CFT.
Now we are going to count the degrees of freedom for gravity theory in AdS$_{d+1}$

spacetime. We know that the number of the degrees of freedom is captured by the horizon area which reads

$$N_{dof}^{AdS} = \frac{A_H}{4G_N}.$$

The metric in AdS_{d+1} reads

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + dz^2 + \sum_{i=1}^{d-1}(dx^i)^2\right] . \tag{2.127}$$

So the horizon area is

$$A_H = \int d^{d-1}x\sqrt{g} \ = \left(\frac{LR}{\epsilon}\right)^{d-1} . \tag{2.128}$$

We know that $G_N \sim L_p^{d-1}$ where L_p is the Planck length. Using this fact we obtain

$$N_{dof}^{AdS} = \left(\frac{RL}{\epsilon L_p}\right)^{d-1} \tag{2.129}$$

From equivalence of two theories, we know that $N_{dof}^{CFT} = N_{dof}^{AdS}$ which implies

$$c = \frac{1}{4}\left(\frac{R}{L_p}\right)^{d-1} . \tag{2.130}$$

For classical gravity we must say from above expression $c >> 1$.

2.7 Aspects of gauge/gravity duality

There are many aspects of gauge/gravity duality. The main aspect of this duality is to analyze the physics of strongly coupled system with help of weakly coupled gravitational system. The AdS/CFT dictionary gives us a precise map between the quantities in gravity theory and the quantities in strongly coupled gauge theory. The phase transition in gravity theory will help us to understand the phase transition in strongly coupled gauge theory with help of this AdS/CFT dictionary. The dual theory of Hawking-Page phase transition is associated with confinement/de-confinement phase transition in QCD. The dual theory of second order phase transition in gravity theory is associated with phase transition in strongly coupled superconductors. This is one of the main aspects of this duality. Besides this, entanglement entropy of strongly coupled field theory and hydrodynamics can be described by investigating weakly coupled gravitational system.

2.7.1 Phase transition

Phase is the state of matter which is described by different macroscopic properties. There are different states like liquid, gas,solid and superconducting state. Depending upon temperature of the system, the phase of system changes in another phase.

In thermodynamics, there are different kinds of phase transition, namely, first order phase transition and second order phase transition etc. These two phase transitions are mainly observed in nature, which are described by discontinuity of the derivative of free energy of the system. The first derivative of free energy of a system is discontinuous in a phase transition, called first order phase transition. For first order phase transition, the entropy is discontinuous. Phase transition from water to steam is an example of first order phase transition. For second order phase transition, the second derivative of free energy is discontinuous and the entropy is continuous. Phase transition from metal to superconductor is an example of second order phase transition. Phase transition is due to spontaneous symmetry breaking which is related to order parameter. From Ehrenfest's classification scheme, the order of a transition is the order of the lowest derivative of Gibbs free energy G which shows a discontinuity.
First order phase transition: $S = -\left(\frac{\partial G}{\partial T}\right)_P$ and $V = -\left(\frac{\partial G}{\partial P}\right)_T$ are discontinuous.
Second order phase transition: $C_P = -T\left(\frac{\partial^2 G}{\partial T^2}\right)$ is discontinuous.
where C_P, V, P are the specific heat, volume and pressure of the system respectively.

The first order phase transition in gravity theory was shown first by Hawking and Page in [70]. They considered AdS-Schwarzschild black hole geometry with spherical symmetry which reads

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \\ f(r) &= \left(1 + \frac{r^2}{L^2} - \frac{r_0^3}{L^2 r}\right) \end{aligned} \tag{2.131}$$

where $r_0^3 = 2GM$. Using this metric they showed that this geometry changes to thermal AdS spacetime. The physical interpretation is that black hole evaporates and can shrink completely to photon gas by Hawking radiation. We know that black hole has entropy. But thermal AdS spacetime does not have any entropy. Therefore there is a discontinuity in entropy when the AdS-Schwarzschild black hole changes to thermal AdS spacetime. This tells us that Hawking-Page phase transition is first order phase transition.

In gauge/gravity duality, we frequently use AdS_5-Schwarzschild black hole with planner symmetry which reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2) \tag{2.132}$$

$$f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_h^4}{r^4}\right) \tag{2.133}$$

where r_h is horizon radius. Using gauge/gravity duality, it was shown in [14] that this geometry is dual to the $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory in 4-dimensional spacetime which has only plasma phase. Therefore we say that plasma phase of quantum chromodynamics (QCD) can be described by a gravity theory with AdS_5-Schwarzschild black hole geometry. QCD is a Yang-Mills theory which describes only confining phase, it fails to describe deconfining phase. Introducing

supersymmetry to QCD, it also is difficult to describe deconfining phase because perturbation does not work in strong coupling limit. The gauge/gravity duality allows us to study such strongly coupled system using weakly coupled gravity theory. Studying black hole phase transition, we try to find new insights in confining phase to deconfining phase transition in QCD. Investigating free energy of this spacetime, one can show that this black hole spacetime changes to thermal AdS spacetime which is considered as dual to confining phase of QCD.
Setting $\hbar = 1$, the Hawking temperature for AdS_5-Schwarzschild black hole reads

$$T_H = \frac{f'(r_h)}{4\pi} = \frac{r_h}{\pi L^2} \tag{2.134}$$

where $r_h^4 = 2G_5 M$. This tells us that $T_H \propto M^{\frac{1}{4}}$ which leads to specific heat of the black hole

$$C_H = \frac{\partial M}{\partial T_H} \propto M^{\frac{3}{4}} \tag{2.135}$$

which is positive value. This implies that black hole in AdS spacetime with planar symmetry behaves like ordinary statistical object in which specific heat is always positive. To calculate entropy of this planar black hole, we must add a cut off, L_c, to avoid divergent. The spatial extension of black hole are $0 \leq x, y, x \leq L_c$ which are also spatial coordinate of gauge theory which lives on the boundary of this black hole spacetime. The volume of gauge theory is $V_3 = L_c^3$ which is related to horizon area of this geometry (2.133) in following way

$$A = \int_0^{L_c} dx \int_0^{L_c} dy \int_0^{L_c} dz \left(\frac{r_h}{L}\right)^3 = \left(\frac{r_h}{L}\right)^3 V_3 \tag{2.136}$$

For this black hole, the entropy is

$$\begin{aligned} S_{BH} = \frac{A}{4G_5} &= \frac{V_3}{4G_5}\left(\frac{r_h}{L}\right)^3 \\ \Rightarrow S_{BH} &= \frac{V_3 \pi^3 L^3}{4G_5} T_H^3 \ . \end{aligned} \tag{2.137}$$

From first law of thermodynamics, we find

$$\begin{aligned} dE &= T_H dS_H \\ E &= \frac{3V_3 \pi^3 L^3}{16 G_5} T_H^4 \ . \end{aligned} \tag{2.138}$$

From AdS/CFT dictionary we know that $N^2 = \frac{\pi L^3}{2G_5}$ where N represent number of color in gauge theory. Using this identity and the dictionary, we find energy density

$$\tilde{E} = \frac{E}{V_3} = \frac{3\pi^2}{8} N^2 T^4 \sim \mathcal{O}(N^2) \tag{2.139}$$

where T is the temperature of gauge theory.

The metric for thermal AdS_5 spacetime reads

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2}{r^2}dr^2 \tag{2.140}$$

in which one can define a temperature with Euclidean version of this metric ($t = it_E$) and following identification

$$t_E \to t_E + \beta \tag{2.141}$$

where β is a parameter of this metric. Since there is no horizon in this metric, one have to follow on-shell action formalism for calculating the energy density of this spacetime. The energy density of this spacetime is independent of the number of color N in gauge theory. From free energy analysis, one can show that the phase transition from the black hole geometry to thermal AdS spacetime is determined by a critical temperature T_c. Below this T_c, thermal AdS spacetime dominates and AdS_5-Schwarzschild black hole dominates above the T_c.

From field theory side, one can show that energy density of $SU(N)$ gauge theory can be expressed in terms of the number of color of the gauge theory. The dominated part of partition function of the gauge theory in low temperature limit and high temperature limit respectively,

$$\begin{aligned} \mathcal{Z} &\sim e^{\#\mathcal{O}(N^0)} \quad \text{for low temperature} \\ \mathcal{Z} &\sim e^{\#\mathcal{O}(N^2)} \quad \text{for high temperature} \end{aligned} \tag{2.142}$$

Using the dictionary, the critical temperature in gravity theory is considered as the temperature of gauge theory in which one can distinguish the partition function in above way. Comparing with eq.(2.139) and the above eq.(2.142), we can conclude that AdS_5-Schwarzschild black hole spacetime is dual to $\mathcal{N} = 4$ SYM theory which has no confined phase. The thermal AdS geometry is dual to confined phase. This is the dual interpretation of Hawking-Page phase transition in gauge theory.

For second order phase transition, the change of entropy is continuous which tells us that we need to find a phase transition from a black hole geometry to another black hole geometry. It is not trivial because of no hair theorem. Gubser first showed that the scalar hair formation outside a black hole is possible for asymptotic AdS spacetime. This leads to one clue that phase transition from a black hole geometry with scalar hair to black hole geometry without scalar hair is a possible second order phase transition in gravity theory. We have given details of this phase transition in gravity theory in holographic superconductor chapter.

2.7.2 Hydrodynamics & Quantum entanglement

Before concluding this chapter, for the sake of completeness we would like to briefly mention about shear viscosity to entropy density ratio $\frac{\eta}{s}$ of superfluid and entanglement entropy (EE) from the gauge/gravity perspective. We would like to point out that our works in this direction are not included in this thesis.

We start our discussion with hydrodynamics. In the presence of dissipation, fluids are associated with viscosity which is a measure of internal friction in a fluid. In field theory framework, this viscosity is associated with the transport of momentum. These transport properties of strongly coupled field theories now have been analyzed using gauge/gravity correspondence. From Relativistic Heavy Ion Collider (RHIC) experiment [96, 97], we know that viscosity of superfluid (likely ideal fluid) has a small non-zero value. This indicates that viscosity of a fluid at a finite temperature always has lower bound. This bound is hard to describe by conventional method. Using gravity model of fluid dynamics, the ratio between the shear viscosity and the entropy density has been found as the universal value $\frac{1}{4\pi}$ [14] (lower bound). In gravity model, there are two type of black hole, namely, extremal black hole and non-extremal black hole which play an important role in the computation of this ratio. The black hole with non-zero Hawking temperature are known as non-extremal black hole. Using Einstein gravity [14]-[20], Gauss Bonnet gravity [98]-[101], this ratio has been also investigated for different type of electrodynamics, namely, Maxwell electrodynamics [102] and Power law electrodynamics [103]. The universal value of the ratio has been modified by positive correction as well negative correction for non-extremal black hole cases. For non-extremal case, the $\frac{\eta}{s}$ ratio has been investigated for GB gravity coupled with Born-Infeld electrodynamics in our work [21]. The effects of non-linearity due to non-linear electrodynamics in the ratio of the shear viscosity to the entropy density has been analyzed.

Entanglement entropy (EE) is a fundamental quantity in quantum information theory as it provides a measure for quantum correlation in a bipartite quantum system. In conformal field theory, the EE between a sub-region $\mathcal{A}$ and its complement can be computed through replica trick [79]. Using the gauge/gravity correspondence, the computation of the EE of a boundary CFT can be done by bulk gravitational dual theory. This insight comes the holographic principle which states that the number of degrees of freedom in a region of space is equal in number to the degrees of freedom on the boundary that surrounds the space. In holographic, the EE is given by the area of minimal bulk surface with the same boundary as $\mathcal{A}$, divided by $4G_N$ [22], which is a very simple way to compute the EE. The result from gravity model matches with the result from field theory, which allows us to study general properties of quantum entanglement [23],[28]. In this context we have investigated the entanglement thermodynamics for d-dimensional charged AdS black hole in [29] by studying the holographic entanglement entropy in different cases extremal and non-extremal cases in two different regimes, namely, the low temperature and high temperature limits. A law like the first law of entanglement thermodynamics has also been obtained in the low temperature limit. The analysis helps us to understand the implications of the dimension of spacetime on information theoretic quantities.

Chapter 3

Critical phenomena of higher dimensional holographic superconductors

3.1 Introduction

Holographic superconductor [34],[73]-[76],[104] is a gravity model in one higher spatial dimension to describe high T_c superconductors. Using gauge/gravity duality, this gravity model resembles some basic property of superconductors. This is holographic because we consider one higher dimensional gravity theory. This remarkable connection between condensed matter physics and gravitational physics emerged from gauge/gravity duality. The properties of real world material are described by condensed matter physics which involves gauge degree of freedom whereas gravitational physics describe the astrophysical phenomena. String theory plays a crucial role to establish a map between gauge degree of freedom with the gravitational degree of freedom. Using this mapping we are able to describe basic properties of high T_c superconductors which are difficult to explain using conventional methods in condensed matter physics. To construct the gravitational model describing high T_c superconductors [105], we have to know basics theory of superconductivity which is described briefly in the next section.

3.2 Basics of Superconductivity

The classification of materials according to their transport properties is one of the most important tasks of condensed matter physics. DC conductivity σ which is the static conductivity defined at zero frequency $\omega = 0$, is the quantity to classify the materials as conductors and insulators. One can distinguish metals and insulator through their DC conductivity temperature dependence as the following [106]-[109]: $\frac{d\sigma_{DC}}{dT} < 0 \ \rightarrow$ metal and $\frac{d\sigma_{DC}}{dT} > 0 \ \rightarrow$ insulator. Depending upon the value of DC conductivity, we also distinguish superconductors from other materials. The first theoretical model to describe the DC conductivity of a material is Drude Model

[107]. From this model, it was also shown that the DC conductivity of superconductor is infinite. The electric resistivity of most metals drops suddenly to zero as the temperature is lowered below a critical temperature, this phenomena is called superconductivity. The material which shows this property is called superconductor. Drude model predicts several properties like the Hall effect, electric and thermal frequency with great accuracy. This model fails to describe the different temperature dependency of conductivity in different type of materials. To resolve the limitation of Drude model, the idea of quantum nature of electron play crucial role which is shown in Sommerfeld model [110]. To improve more features in a theoretical model of solid, the Bloch theorem and Kronig-Penney model [111] play an important role which show the formation of band structure and energy gap between two bands, namely, valence band and conduction band. The band structure [106] is a good description to distinguish metal, insulator and semiconductor in a fundamental way. We are now going to discuss the basic phenomenological model of superconductivity [112],[113].

In 1911, Kamerlingh Onnes [114] noticed that the resistivity of Hg metal vanished abruptly at about 4K. Later on, several materials were discovered with zero resistivity property at very low temperature. An external magnetic field is expelled by these materials below the critical temperature which is known as Meissner effect [115]. Based on the magnetic response of superconductors, they are classified as Type-I and Type-II superconductors [107],[112]. The formation of vortex is seen only in TypeII superconductors. Depending upon coupling value between two electrons, we classify them into two categories, namely, weakly coupled superconductors (which are generally low $T_c < 30K$ superconductors) and strongly coupled superconductors (which are generally high $T_c > 30K$ superconductors). It was found that conductivity of barium-doped compound of lanthanum and copper oxide was infinite below the critical temperature $T_c = 35K$ in 1986 [105] which is high T_c superconductor.
In summary, four fundamental properties are seen in a superconductor :

- Infinite conductivity and resistivity $\rho = 0$
- Meissner effect and flux quantization $\phi = n\frac{hc}{2e}$
- Second order phase transition
- Critical magnetic field

For perfect diamagnetism implies $\chi = \frac{M}{H} = -\frac{1}{4\pi}$ which is obtained from $B = H + 4\pi M$ by putting $B = 0$. Specific heat is given by $C_V = \gamma T + \beta T^3$. Perfect diamagnetism is corresponding to perfect conductor $\rho = 0$ which does not explain the Meissner effect. The superconducting state is shown in 3.1.

3.2.1 Phenomenological Models

Superfluid model [116] is a phenomenological theory of superconductivity which is based on the two fluid model. The fluid of electrons in superconducting state consists

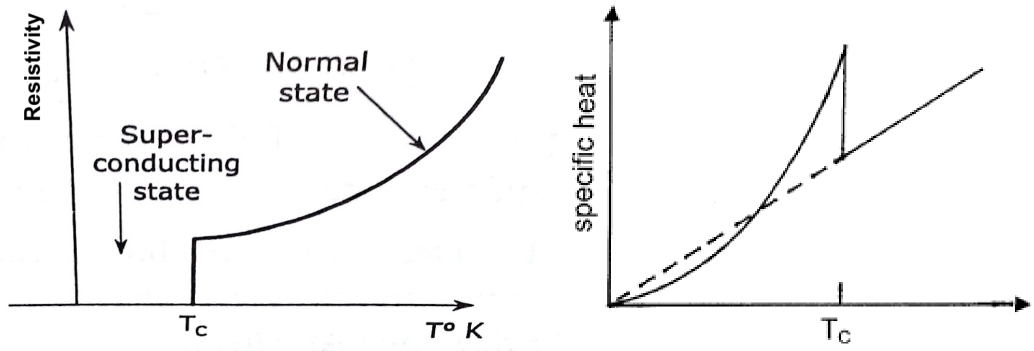

Figure 3.1: Resistivity vs. temperature and specific heat vs. temperature plots of a superconductor

of electrons of normal fluid state and electrons of superfluid state. The total number of electrons n in the superconducting state is given by

$$n = n_s + n_n \tag{3.1}$$

where n_s and n_n represent number of superfluid electron and number of normal fluid electron respectively. There are three phenomenological models based on two fluid model [112] :

- Gorter-Casimir Model
- London-Pippard Theory
- Ginzburg-Landau Theory

(I) Gorter-Casimir two fluid model : This model is based on thermodynamic response [117]. The free energy of superconducting state is given by

$$F_s(x,T) = \sqrt{x} f_n(T) + (1-x) f_s(T) \tag{3.2}$$

where $x = \frac{n_n}{n}$ is normal fluid electron fraction and $f_n(T)$ is free energy density in normal state and $f_s(T)$ is free energy density in superconducting state. We know that $f_n(T) = -\frac{1}{2}\gamma T^2$ for normal state where γ is the electronic specific heat coefficient. We choose $f_s(T) = \beta$(constant) and substitute in above equation

$$F_s(x,T) = -\frac{\sqrt{x}}{2}\gamma T^2 + (1-x)\beta \ . \tag{3.3}$$

The phase transition is occurred due to the discontinuity of derivative of free energy which implies

$$\frac{\partial F_s}{\partial x} = 0 \quad \Rightarrow \quad x_{eq.} = \frac{\gamma^2 T^4}{16\beta^2} = \frac{T^4}{T_c^4} \tag{3.4}$$

where $T_c = 2\sqrt{\frac{\beta}{\gamma}}$ is defined as the critical temperature. For $T = 0$ and $T = T_c$, this normal fluid fraction becomes $x_{eq.}(0) = 0$ and $x_{eq.}(T_c) = 1$ which is expected from definition of $x = \frac{n_n}{n}$. In this equilibrium state, we find $\beta = \frac{1}{4}\gamma T_c^2$. The free energy in normal state is given by

$$F_n = f_n(T) = -\frac{1}{2}\gamma T^2 \ . \tag{3.5}$$

Substituting $x_{eq.}$ and β in eq.(3.3), we obtain the difference between free energy of superconducting state and normal state at equilibrium

$$\begin{aligned} F_s(T) - F_n(T) &= -\frac{\gamma T^4}{2T_c^2} - \left(1 - \frac{T^4}{T_c^4}\right)\frac{1}{4}\gamma T_c^2 + \frac{1}{2}\gamma T^2 \\ &= -\frac{\gamma T_c^2}{4}\left[1 - \frac{T^2}{T_c^2}\right]^2 \\ &= -\frac{H_c^2(T)}{8\pi} \end{aligned} \tag{3.6}$$

where

$$H_c(T) = H_0\left[1 - \frac{T^2}{T_c^2}\right] \quad : \quad H_0 = \sqrt{2\pi\gamma T_c^2} \ . \tag{3.7}$$

The H_c is called the critical magnetic field in which superconductivity of a material is destroyed. The electronic specific heat is defined by

$$\begin{aligned} C_V^s \equiv -T\left(\frac{\partial^2 F_s}{\partial T^2}\right)_V &= -T\frac{\partial^2}{\partial T^2}\left\{-\frac{\gamma T^4}{2T_c^2} - \left(1 - \frac{T^4}{T_c^4}\right)\frac{1}{4}\gamma T_c^2\right\} \\ &= 3\gamma T_c\left(\frac{T}{T_c}\right)^3 \ . \end{aligned} \tag{3.8}$$

At $T = T_c$, the electronic specific heat of superconductor takes value $C_V^e(T_c) = 3\gamma T_c$ which does not agree with experimental findings (like $e^{-\Delta/T}$). This is the first failure of this model. We know that the electronic specific heat for normal phase is $C_V^n = \gamma T_c$. Now we calculate $\frac{\Delta C_V}{C_V^n} = \frac{C_V^s - C_V^n}{C_V^n} = 2$ which does not agree with experimental value 1.43. This is the second failure of this model.

(II) London-Pippard Theory : This theory is based on magnetic response. London [118] introduced the total current density in superconducting state as sum of current density of normal electron (j_n) and current density of superfluid electron (j_s)

$$j = j_s + j_n \tag{3.9}$$

with assumption $j_s >> j_n$. We know that $n = n_s + n_n$. The current density for superfluid electron can be expressed as

$$j_s = -n_s e v_s \tag{3.10}$$

where v_s is drift velocity of superfluid electron and e is charge of electron. The motion of electron in applied electric field is governed by Newton second law which reads

$$m_e\frac{d\vec{v}_s}{dt} = -e\vec{E} \ . \tag{3.11}$$

Using eq.(3.11), we obtain from eq.(3.10)

$$\frac{d\vec{j}_s}{dt} = \frac{n_s e^2}{m_e}\vec{E} \tag{3.12}$$

which is called the first London equation. Now we take curl of the eq.(3.12) and use the Maxwell equation, we get

$$\nabla \times \vec{j}_s = -\frac{n_s e^2}{m_e}\vec{B} \tag{3.13}$$

which is called the second London equation. Using Ampere law $\nabla \times \vec{B} = 4\pi\vec{j}_s$, we find from eq.(3.13)

$$\nabla^2\vec{B} = \frac{1}{\lambda_L^2}\vec{B} \tag{3.14}$$

where $\lambda_L = \sqrt{\frac{m_e}{4\pi n_s e^2}}$ is called penetration length. When magnetic field is along x-direction only, the solution of magnetic field in a superconductor is $B(x) = B(0)e^{-\frac{x}{\lambda_L}}$. Using fact that $n_s = n\left(1 - \frac{T^4}{T_c^4}\right)$, we find

$$\lambda_L(T) = \frac{\lambda_L(0)}{\sqrt{1 - \frac{T^4}{T_c^4}}} \tag{3.15}$$

where $\lambda_L(0) = \sqrt{\frac{m_e}{4\pi n e^2}}$. We know that $B = \nabla \times \vec{A}$. Substituting it in eq.(3.13), we find

$$\vec{j}_s(r) = -\frac{n_s e^2}{m_e}\vec{A}(r) \tag{3.16}$$

which is the common form of London equation [112]. This is a local equation. This theory does not describe the dependency of impurities in λ. To incorporate the quantum signature, Pippard [119] introduced non-locality in London's equation in the following manner

$$\vec{j}_s(r) = -\int_{|r-r_0|<\xi_0} K(r,r')\vec{A}(r').d\vec{r'} \tag{3.17}$$

where $K(r,r')$ is propagator and ξ_0 define quantum region which is called coherence length. In normal state ξ_0 is the mean free path. Now the theory consists of two length scale (λ_L, ξ_0) with flux quantization $\phi_n = \frac{nhc}{e}$ which contradicts with experimental findings $\phi_n = \frac{nhc}{2e}$. If we consider effective charge of an electron in superconducting state as $e* = 2e$, then theoretical finding matches with experimental value.

(III) Ginzburg-Landau Theory : This is a phenomenological theory for second order phase transition in which free energy is expressed in power series of order parameter. Minimization of free energy with respect to order parameter gives rise to second order phase transition. The difference between free energy of superconducting phase and normal phase which is expressed as [120]

$$F(\psi, T) = a(T)|\psi|^2 + b(T)|\psi|^4 + c(T)|\psi|^6 + \cdots \tag{3.18}$$

where $|\psi|^2$ is the order parameter. This order parameter is defined by $|\psi(r)|^2 = \frac{n_s(r)}{n}$ which is local electron density in condensed state. Since n_s is independent of r for homogeneous case, ψ is independent of r. As $T \to T_c$, $|\psi|^2 << 1$, we consider only first two terms in the above expression. By minimization of free energy with respect to order parameter $|\psi|^2$, we find

$$|\psi_e|^2 = -\frac{a(T)}{2b(T)} \ . \tag{3.19}$$

The minimum value of the difference between free energy of superconducting state and normal state is

$$F_{min} = -\frac{a^2(T)}{4b(T)} \ . \tag{3.20}$$

In Gorter-Casimir two fluid model, we know from eq.(3.6) that the difference between free energy of superconducting state and normal state is $F_s - F_n = -\frac{\gamma T_c^2}{4}\left[1 - \frac{T^2}{T_c^2}\right]$. Comparing with eq.(3.20), we find that

$$\frac{a^2(T)}{4b(T)} = \frac{\gamma T_c^2}{4}\left[1 - \frac{T^2}{T_c^2}\right]^2 \ . \tag{3.21}$$

We also know that $n_s \propto \frac{1}{\lambda_L^2}$ from definition of λ_L. This implies that the order parameter is also inversely proportional to $\lambda_L(T)$. From eq.(3.19), we find that

$$\frac{a(T)}{2b(T)} = -\frac{\lambda_L^2(0)}{\lambda_L^2(T)} = \left(1 - \frac{T^4}{T_c^4}\right) \tag{3.22}$$

using the fact that $|\psi_e(0)|^2 = 1$. Solving eq.(3.21) and eq.(3.22), we obtain

$$a(T) = \frac{\gamma}{2}T_c(T - T_c) \tag{3.23}$$

$$b(T) = \frac{\gamma T_c^2}{8} \ . \tag{3.24}$$

We now substitute $a(T)$ and $b(T)$ in free energy expression and obtain

$$\begin{aligned} F(\psi, T) &= \frac{\gamma}{2}T_c(T - T_c)|\psi|^2 + \frac{\gamma T_c^2}{8}|\psi|^4 \\ &= \alpha(T - T_c)|\psi|^2 + \frac{\beta}{2}|\psi|^4 \end{aligned} \tag{3.25}$$

where α and β are positive constants. Free energy minimization with respect to order parameter $|\psi|^2$ gives

$$\frac{dF(\psi, T)}{|d\psi|^2} = 0 \to \psi = 0 \quad or \quad \psi^2 = -\frac{\alpha}{\beta}(T - T_c) \ . \tag{3.26}$$

For $\psi = 0$

$$F'' = \left.\frac{d^2F(\psi, T)}{d(|\psi|^2)^2}\right|_{\psi=0} = 2\alpha(T - T_c) \tag{3.27}$$

which is minimum ($F'' > 0$) when $T > T_c$. For $\psi^2 = -\frac{\alpha}{\beta}(T - T_c)$

$$F'' = \frac{d^2F(\psi, T)}{d(|\psi|^2)^2} = -4\alpha(T - T_c) \tag{3.28}$$

which is minimum ($F'' > 0$) when $T < T_c$. So we see that free energy is minimum for finite value of $|\psi|^2 = -\frac{\alpha}{\beta}(T - T_c)$ which is associated with breaking of $U(1)$ symmetry. The minimum free energy is

$$F_{min} = -\frac{\alpha^2}{2\beta}(T - T_c)^2 \ . \tag{3.29}$$

Using this $U(1)$ spontaneous symmetry breaking, the second order phase transition in superconductor is explained. There are two ways to see the spontaneous symmetry breaking.
1. Hamiltonian does break symmetry to define ground state of a system.
2. Hamiltonian have symmetry, but ground state does not have that symmetry.
The superfluid state is defined by

$$\langle b(r)\rangle = |\psi(r)|^2 \tag{3.30}$$

which is the order parameter. For this operator, the superfluid state $\langle b(r)\rangle$ has non-vanishing value. In normal phase $\langle b(r)\rangle = 0$. These two phase are defined by the critical temperature of the system in which $\langle b(r)\rangle = 0$ above T_c and $\langle b(r)\rangle \neq 0$ below T_c.

3.2.2 BCS theory of superconductivity

Bardeen–Cooper–Schrieffer theory [121] (namely BCS theory) is the first microscopic theory of superconductivity. It describes microscopic effect caused by a condensation of Cooper pairs into a boson-like state. The boson-like behaviour of such electron pairs are called "Cooper pairs". The condensation of Cooper pairs is the foundation of the BCS theory of superconductivity. A passing electron attracts the lattice, causing a slight ripple toward its path. Cooper [122] showed that an arbitrarily small attraction between electrons in a metal can cause a paired state of electrons to have lower energy than the Fermi energy, which implies that the pair is bound via phonon interaction.
If superconductivity property is only due to electrons, then the critical temperature T_c should be independent on the mass of the nucleons. Experiments show that the critical temperature T_c depends on the mass of the nucleons. This shows a hint that superconductors are related to lattice motions (phonons). An electron attracts nearby nucleons and its mass is much smaller than nucleon mass. So electrons move much faster than lattices. When electron flies away, the lattice distortions will not

recover immediately. So, there is locally a higher density of positive charges, which will attract other electrons. In this way, an electron will attract other electrons, via lattice distortions. In quantum field theory, it was shown that this attraction is induced when two electrons exchange a virtual phonon.
The BCS theory is based on the Fermi liquid model with attractive interaction of electrons. In the form of second quantization, the Hamiltonian is [123]

$$H = \sum_{k,\sigma} \epsilon(k) c^\dagger_{k\sigma} c_{k\sigma} + \frac{1}{2} \sum_{\sigma,\sigma'} \sum_{r,r'} V(r-r') c^\dagger_{r\sigma} c_{r\sigma} c^\dagger_{r'\sigma'} c_{r'\sigma'} \tag{3.31}$$

so that

$$E = \sum_{k,\sigma} \epsilon(k) n_\sigma(k) + \frac{1}{2} \sum_{\sigma,\sigma'} \sum_{r,r'} V(r-r') n_\sigma(r) n_{\sigma'}(r') \tag{3.32}$$

where $\epsilon(k) = \langle k|H_0|k\rangle, H_0$ is Hamiltonian without interaction part and $n_\sigma(k)$ is number operator. Transferring the interaction part into k space and simplifying, we get

$$H = \sum_{|k|,\sigma} \epsilon(k) c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,k'} V_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \tag{3.33}$$

For simplicity, we assume that $V_{k,k'} = V$. The Hamiltonian now reads

$$H = \sum_{|k|,\sigma} \epsilon(k) c^\dagger_{k\sigma} c_{k\sigma} + V \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \ . \tag{3.34}$$

In the mean-field approximation, one can replace the operators in the Hamiltonian by their mean values, assuming that the deviations from the mean values are small. Let us write two operators A and B as

$$A = \langle A\rangle + A - \langle A\rangle = \langle A\rangle + \delta A \tag{3.35}$$

$$B = \langle B\rangle + B - \langle B\rangle = \langle B\rangle + \delta B \ . \tag{3.36}$$

Assuming that the fluctuations around the mean values are small, one can neglect the $\delta A \delta B$ term, so we get

$$\begin{aligned} AB &= \langle A\rangle\langle B\rangle + \langle A\rangle\delta B + \langle B\rangle\delta A \\ &= \langle A\rangle B + \langle B\rangle A - \langle A\rangle\langle B\rangle \ . \end{aligned} \tag{3.37}$$

One can generalize this kind of consideration for products of more than two operators (Wick's theorem).
Here, pair of electrons form a Bose-Einstein condensate state, so the order parameter is $\sum_k \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle$. We define $X_{k'} = \langle c_{-k'\downarrow} c_{k'\uparrow}\rangle$ and $X^*_k = \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle$. Considering the operator $A = c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}$ and $B = c_{-k'\downarrow} c_{k'\uparrow}$ and using mean-field approximation, we obtain

$$\begin{aligned} V \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} &= V \sum_{k,k'} \left[X^*_k + (c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - X^*_k)\right] \left[X_{k'} + (c_{-k'\downarrow} c_{k'\uparrow} - X_{k'})\right] \\ &\approx V \sum_{k,k'} \left[X^*_k X_{k'} + (c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - X^*_k) X_{k'} + (c_{-k'\downarrow} c_{k'\uparrow} - X_{k'}) X^*_k\right] \\ &= V \sum_{k,k'} \left[X_{k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + X^*_k c_{-k'\downarrow} c_{k'\uparrow} - X_{k'} X^*_k\right] \end{aligned} \tag{3.38}$$

Putting in the Hamiltonian, we find

$$\begin{aligned} H_{MF} &= \sum_k [\epsilon(k) c^\dagger_{k\uparrow} c_{k\uparrow} + \epsilon(k) c^\dagger_{k\downarrow} c_{k\downarrow}] + V \sum_k X^*_k \sum_{k'} c_{-k'\downarrow} c_{k'\uparrow} \\ &+ V \sum_{k'} X_{k'} \sum_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - V \sum_{kk'} X^*_k X_{k'} \end{aligned} \tag{3.39}$$

Defining $\Delta = V \sum_k X_k$ and $\Delta^* = V \sum_k X^*_k$, we get

$$H_{MF} = \sum_k \left[\epsilon(k) c^\dagger_{k\uparrow} c_{k\uparrow} + \epsilon(k) c^\dagger_{k\downarrow} c_{k\downarrow} \right] + \Delta^* \sum_k c_{-k\downarrow} c_{k\uparrow} + \Delta \sum_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - \frac{\Delta^* \Delta}{V} . \tag{3.40}$$

We know that $N = \sum_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} = \sum_k \left[c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow} \right]$. For fixed chemical potential μ, it makes more sense to consider $H - \mu N$, instead of H.

$$\begin{aligned} H_{MF} - \mu N &= \sum_k \left[\{\epsilon(k) - \mu\} c^\dagger_{k\uparrow} c_{k\uparrow} + \{\epsilon(k) - \mu\} c^\dagger_{k\downarrow} c_{k\downarrow} + \Delta^* c_{-k\downarrow} c_{k\uparrow} + \Delta c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right] - \frac{\Delta^* \Delta}{V} \\ &= \sum_k \begin{bmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{bmatrix} \begin{bmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{bmatrix} \begin{bmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{bmatrix} - \frac{\Delta^* \Delta}{V} \end{aligned} \tag{3.41}$$

where $\xi_k = \epsilon(k) - \mu$.
This is similar to Hamiltonian of a 2-band systems. Now it can be diagonalized by Bogoliubov transformation which is defined as

$$\gamma_{1k} = u c_{k\uparrow} + v c^\dagger_{-k\downarrow} \qquad \& \qquad \gamma^\dagger_{1k} = u^* c^\dagger_{k\uparrow} + v^* c_{-k\downarrow} \tag{3.42}$$

$$\gamma_{2k} = -v^* c_{k\uparrow} + u^* c^\dagger_{-k\downarrow} \qquad \& \qquad \gamma^\dagger_{2k} = -v c^\dagger_{k\uparrow} + u c_{-k\downarrow} \tag{3.43}$$

provided $|u|^2 + |v|^2 = 1$. So the diagonal basis can be written in the following form

$$\begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} = \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix} \begin{bmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{bmatrix} . \tag{3.44}$$

The propertics of γ_{ik} can be analysed by studying commutation relation. From the definition of Bogoliubov transformation, we can find

$$\{\gamma^\dagger_{1k}, \gamma_{1k}\} = u^2 + v^2 = 1 = \{\gamma^\dagger_{2k}, \gamma_{2k}\} \qquad \& \qquad \{\gamma^\dagger_{1k}, \gamma_{2k}\} = 0 . \tag{3.45}$$

They represent fermions since they follow anticommutation relation. We now write down the anihalation and creation operators in terms of diagonal basis as

$$\begin{aligned} \begin{bmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{bmatrix} &= \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} \\ &= \begin{bmatrix} u^* & -v \\ v^* & u \end{bmatrix} \begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} \end{aligned} \tag{3.46}$$

and dual of this takes form as

$$\begin{bmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{bmatrix} = \begin{bmatrix} \gamma^\dagger_{1k} & \gamma^\dagger_{2k} \end{bmatrix} \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix} . \tag{3.47}$$

Substituting above in eq.(3.41), we find

$$\begin{aligned} H_{MF} - \mu N &= \sum_k \begin{bmatrix} \gamma_{1k}^\dagger & \gamma_{2k}^\dagger \end{bmatrix} \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix} \begin{bmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{bmatrix} \begin{bmatrix} u^* & -v \\ v^* & u \end{bmatrix} \begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} - \frac{\Delta^*\Delta}{V} \\ &= \sum_k \begin{bmatrix} \gamma_{1k}^\dagger & \gamma_{2k}^\dagger \end{bmatrix} \mathbb{M} \begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} - \frac{\Delta^*\Delta}{V} \end{aligned} \tag{3.48}$$

where

$$\mathbb{M} = \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix} \begin{bmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{bmatrix} \begin{bmatrix} u^* & -v \\ v^* & u \end{bmatrix} = \begin{bmatrix} \xi u^2 + \Delta v^* u + \Delta^* u^* v - \xi v^2 & -2\xi uv + \Delta u^2 - \Delta^* v^2 \\ -2\xi u^* v^* - \Delta v^2 + \Delta^* u^2 & \xi v^2 - \Delta u v^* - \Delta^* u * v - \xi u^2 \end{bmatrix}.$$

But the matrix $\mathbb{M}$ should be diagonal form since γ_{ik} is diagonal basis. This implies $\mathbb{M}_{12} = 0 = \mathbb{M}_{21}$ which leads to

$$2\xi uv = \Delta u^2 - \Delta^* u^2 \ . \tag{3.49}$$

We now define

$$u = e^{i\varphi_u} \cos\eta \quad , \quad v = e^{i\varphi_v} \sin\eta \quad , \quad \Delta = |\Delta| e^{i\varphi} \tag{3.50}$$

to simply $\mathbb{M}_{11}$ and $\mathbb{M}_{22}$. Substituting in eq.(3.49), we get

$$2\xi e^{i(\varphi_u+\varphi_v)} \cos\eta \sin\eta = |\Delta| \left\{ e^{i(2\varphi_u+\varphi)} \cos^2\eta - e^{i(2\varphi_v-\varphi)} \sin^2\eta \right\} \ . \tag{3.51}$$

We demand that phase factor of both side of above equation is equal, which leads to following conditions

$$\varphi_u - \varphi_v = -\varphi \qquad \Rightarrow \qquad \boxed{\varphi_u = -\varphi_v = -\frac{\varphi}{2}} \tag{3.52}$$

Using this condition, we find

$$2\xi \cos\eta \sin\eta = |\Delta| \left\{ \cos^2\eta - \sin^2\eta \right\} \qquad \Rightarrow \qquad \tan 2\eta = \frac{|\Delta|}{\xi} \ . \tag{3.53}$$

Using eq.(3.50) and above relation, we find

$$\begin{aligned} \mathbb{M}_{11} &= \xi(u^2 - v^2) + \Delta v^* u + \Delta^* u^* v \\ &= \xi \cos 2\eta + |\Delta| \sin 2\eta \\ &= \sqrt{\xi^2 + |\Delta|^2}. \end{aligned} \tag{3.54}$$

Similarly we find $\mathbb{M}_{22} = -\sqrt{\xi^2 + |\Delta|^2}$. So the diagonalized form of $\mathbb{M}$ is

$$\mathbb{M} = \begin{bmatrix} \sqrt{\xi^2 + |\Delta|^2} & 0 \\ 0 & -\sqrt{\xi^2 + |\Delta|^2} \end{bmatrix} \tag{3.55}$$

Substituting this in eq.(3.48) one can obtain

$$H_{MF} = \mu N + \sum_k \begin{bmatrix} \gamma_{1k}^\dagger & \gamma_{2k}^\dagger \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & -E_k \end{bmatrix} \begin{bmatrix} \gamma_{1k} \\ \gamma_{2k} \end{bmatrix} - \frac{\Delta^*\Delta}{V} \tag{3.56}$$

where $E_k = \sqrt{\xi^2 + |\Delta|^2}$.
For the top band, we have energy $E_k \geq |\Delta|$ and for the bottom band, we have energy $E_k \leq -|\Delta|$. So, the energy gap between two band is $2|\Delta|$, if $|\Delta| > 0$. One can calculate the value of Δ by self-consistent condition. For $T > T_c$, there is only one solution of self-consistency equations, $\Delta = 0$. For $T < T_c$, we have one solution $\Delta = 0$, which corresponds to the unstable solution for Ginzburg–Landau free energy. Here we have also a solution with $|\Delta| = \Delta(T)$.
Self-consistent approach : The value of Δ is determined by self-consistent condition. From definition of Δ, we find

$$\begin{aligned}\Delta &= V\sum_k X_k = V\sum_k \langle c_{-k\downarrow}c_{k\uparrow}\rangle = V\sum_k \langle\left(v\gamma_{1k}^\dagger + u^*\gamma_{2k}^\dagger\right)\left(u^*\gamma_{1k}^\dagger - v\gamma_{2k}^\dagger\right)\rangle \\ &= V\sum_k \left\{u^*v\langle\gamma_{1k}^\dagger\gamma_{1k}\rangle - u^*v\langle\gamma_{2k}^\dagger\gamma_{2k}\rangle\right\} \end{aligned} \tag{3.57}$$

Since γ_{ik} represents fermion, they follow Fermi-Dirac statistics and

$$\langle\gamma_{1k}^\dagger\gamma_{1k}\rangle = \frac{1}{e^{\frac{E_k}{k_BT}}+1} \quad , \quad \langle\gamma_{2k}^\dagger\gamma_{2k}\rangle = \frac{1}{e^{\frac{-E_k}{k_BT}}+1} \tag{3.58}$$

Using eq.(3.58) and eq.(3.50), we find

$$\begin{aligned}|\Delta|e^{i\varphi} &= Ve^{i(\varphi_v-\varphi_u)}\sum_k \cos\eta\sin\eta\left\{\frac{1}{e^{\frac{E_k}{k_BT}}+1} - \frac{1}{e^{\frac{-E_k}{k_BT}}+1}\right\} \\ \Rightarrow |\Delta| &= -\frac{V}{2}\sum_k \frac{|\Delta|}{E_k}\tanh\frac{E_k}{2k_BT} \end{aligned} \tag{3.59}$$

which is called self-consistent equation for $|\Delta|$. $\Delta = 0$ is always solution for this equations which implies that there is only one solution of self-consistency equations for $T > T_c$. For $T < T_c$, we have one solution $\Delta = 0$, which corresponds to the unstable solution for Ginzburg–Landau free energy. Here we have also a solution with $|\Delta| = \Delta(T)$. For $\Delta \neq 0$, we find

$$1 = -\frac{V}{2}\sum_k \frac{1}{E_k}\tanh\frac{E_k}{2k_BT} \tag{3.60}$$

For repulsive interaction, $V = |V|$ which says that eq.(3.60) has no solution since sign of both side in equation is different. For attractive interaction, $V = -|V|$ which allows eq.(3.60) with non-trivial solutions. Near Fermi surface, we can replace $\sum_k$ by $\int d\epsilon_k N(0)$ where $N(0)$ is density of state at zero temperature which is measured directly in scanning tunneling microscope (STM). For attractive interaction, we find

$$1 = \frac{|V|}{2}\sum_k \frac{1}{E_k}\tanh\frac{E_k}{2k_BT} \quad \Rightarrow \quad 1 = \frac{|V|}{2}\int d\epsilon_k \frac{N(0)}{E_k}\tanh\frac{E_k}{2k_BT} \ . \tag{3.61}$$

We know that the range of interaction is determined by the Debye frequency which is the typical energy scale of the phonon. We now write above integration with

Debye energy limit ϵ_D

$$
\begin{aligned}
1 &= \frac{|V|N(0)}{2}\int_{\mu-\epsilon_D}^{\mu+\epsilon_D} d\epsilon_k \frac{\tanh\frac{\sqrt{(\epsilon_k-\mu)^2+|\Delta|^2}}{2k_BT}}{\sqrt{(\epsilon_D-\mu)^2+|\Delta|^2}} = \frac{|V|N(0)}{2}\int_{-\epsilon_D}^{\epsilon_D} d\xi \frac{\tanh\frac{\sqrt{\xi^2+|\Delta|^2}}{2k_BT}}{\sqrt{\xi^2+|\Delta|^2}} \\
\Rightarrow 1 &= |V|N(0)\int_0^{\epsilon_D} d\xi \frac{\tanh\frac{\sqrt{\xi^2+|\Delta|^2}}{2k_BT}}{\sqrt{\xi^2+|\Delta|^2}} \qquad (3.62)
\end{aligned}
$$

At $T = T_c$, $\Delta = 0$ which implies from eq.(3.62)

$$
\frac{1}{|V|N(0)} = \int_0^{\frac{\epsilon_D}{2k_BT_c}} dx \frac{\tanh x}{x} \qquad (3.63)
$$

where $x = \frac{\xi}{2k_BT_c}$. As we know that $\int_0^b dx\frac{\tanh x}{x} = \ln(\frac{4}{\pi}e^{\zeta}b)$ as $b \to 0$ where $\zeta = 0.57721$ is Euler's constant. In weak coupling limit, ($|V| << \frac{1}{N(0)}$), $k_BT_c << \epsilon_D$ so the upper limit of above integration is very large and integration takes value approximately

$$
\begin{aligned}
\frac{1}{|V|N(0)} &\approx \ln\left(\frac{1.13\epsilon_D}{k_BT_c}\right) \\
\Rightarrow k_BT_c &= 1.13\epsilon_D e^{-\frac{1}{|V|N(0)}}\ . \qquad (3.64)
\end{aligned}
$$

For $T = 0$, we denote $\Delta = \Delta_0$ and we find from eq.(3.62)

$$
1 = |V|N(0)\int_0^{\epsilon_D} d\xi \frac{1}{\sqrt{\xi^2+|\Delta_0|^2}} = |V|N(0)\ln\left(\frac{\epsilon_D+\sqrt{\epsilon_D^2+|\Delta_0|^2}}{|\Delta_0|}\right) \qquad (3.65)
$$

since $\tanh x \to 1$ as $x \to \infty$. In weak coupling limit $|V|N(0) << 1$, we have $|\Delta_0| << \epsilon_D$. Using these conditions, we find

$$
1 \approx |V|N(0)\ln\frac{2\epsilon_D}{|\Delta_0|} \quad \Rightarrow \quad |\Delta_0| = 2\epsilon_D e^{-\frac{1}{|V|N(0)}} \qquad (3.66)
$$

From eq.(3.64) and eq.(3.66), the ratio between the order parameter at zero temperature (Δ_0) and the critical temperature (T_c) for weak coupling limit reads

$$
\boxed{\frac{\Delta_0}{k_BT_c} = \frac{2}{1.13k_B} = 1.764} \qquad (3.67)
$$

which is a universal constant. This is true for all weakly coupled superconductors.

In summary, we can say that the basic properties of low T_c superconductors are explained by this theory. The microscopic mechanism of superconductivity was first explained in details by Barden, Cooper and Schrieffer [121]. The mechanism involves in the formation of pair of two electrons with opposite spin via electron-phonon interaction. This pair behaves like boson particle and they form condensation below the critical temperature. This microscopic theory of superconductors also predicts that the ratio between energy gap and the critical temperature is 1.764 in the weak coupling limit [112],[123]. This value matches with experimental result and it is true for weakly coupled superconductors. There are a new class of high T_c superconductors [105],[124] which show superconductivity along CuO plane only and the ratio between the energy gap and the critical temperature is 3.76 [124]. There are several proposals to understand the mechanism of high T_c superconductor and it is evident that cuprate based superconductor have $d_{x^2-y^2}$ symmetry. The interlayer coupling model proposed that the BCS-type symmetry can be enhanced in a layered structure superconductors by itself. There is evidence that electron pair still form in high T_c superconductors which are strongly coupled superconductor, but the pairing mechanism is not well understood. We try to construct a gravitational model for strongly coupled superconductor in the next section.

3.3 Gravitational dual for superconductors

Gauge/gravity duality is a new tool to study strongly coupled field theory and it allows to compute dynamical transport properties of strongly coupled systems at non-zero temperature. To construct a gravity model for describing superconductivity, we need to find a gravity model which have a notion of temperature and notion of hair formation. The condensation formation is described by scalar hair formation which is a condensate corresponding a matter field outside of a black hole. This implies gravity model should be consist of black hole and some charged scalar field outside of the black hole. Using gauge/gravity duality, Hawking's temperature of a black hole is identified as the temperature of dual field theory which is known as critical temperature. To describe a non-gravitational system, we need to find a black hole which has positive specific heat. The planar black hole in AdS has always positive specific heat. This is not an easy task to find a black hole with scalar hair and positive specific heat. From No-hair theorem, we know that matter field outside of a black hole fall into the horizon or radiate out to infinite for asymptotic flat spacetime. Stable matter field outside of black hole can stay for asymptotic AdS spacetime. This spacetime plays a crucial role for the formation of charged scalar hair and acts like confining box.

Gubser first showed that a charged scalar field around charged black hole in AdS have $U(1)$ spontaneous symmetry breaking property and gave the following gravity model for scalar hair formation in AdS spacetime [33],[125]

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D_\mu\psi|^2 - m^2|\psi|^2\right] \tag{3.68}$$

where $\Lambda = -\frac{3}{L^2}$ is cosmological constant and L is radius of AdS spacetime, $D_\mu = \partial_\mu\psi - iqA_\mu\psi$, A_μ is gauge field, ψ is charged scalar field with charge q. The effective mass of the matter field is

$$m_{eff}^2 = m^2 + q^2\{g^{tt}A_t^2 + g^{ii}A_i^2\} \tag{3.69}$$

which determines the instability of matter field outside the black hole. This leads to the $U(1)$ spontaneous symmetry breaking property of this theory. Without magnetic term ($A_i = 0$), the effective mass becomes $m_{eff}^2 = m^2 + q^2 g^{tt} A_t^2$. The last term is negative and infinite at horizon due to g^{tt} term. So we fix the gauge field at horizon to be zero i.e. $A_t(r_h) = 0$. Outside horizon, it has negative value, so there is a chance m_{eff}^2 becomes sufficiently negative near the horizon to destabilize the scalar field. The black hole develops a scalar hair at low temperature tuning temperature of black hole. Near horizon, the charged particle are created via Schwinger mechanism and they cannot escape and settle outside of the horizon since AdS spactime acts like confining box. In the presence of magnetic field ($A_i \neq 0$), the last term in eq.(3.69) stabilize the normal state since g^{ii} all are positive quantities. So the scalar hair breaks a local $U(1)$ symmetry in bulk, the dual description consists of a second order phase transition in boundary theory. Using gauge/gravity duality and this phase transition in gravity side, the basic properties of high T_c superconductor was first explained in [34] using numerical analysis. It was also shown numerically $\frac{\Delta_0}{k_B T_c} = 4.02$ which agrees with experimental findings (~ 3.78) [124] for high T_c superconductor. More precisely, this description explains the properties of s-wave superconductors [34]-[39], [71]-[76], [126]-[134].

3.4 Holographic superconductors in Born-Infeld electrodynamics with backreaction

A number of studies have been carried out on various holographic superconductor models based on the framework of Maxwell electrodynamics [135]-[139] as well as non-linear electrodynamics, namely, Born-Infeld electrodynamics [140]-[146]. Several properties like critical exponents, condensates have been studied in the framework of Einstein gravity [147, 148] and Gauss-Bonnet gravity [127, 130, 136, 139, 143, 144, 149, 150] which takes into account the effect of higher curvature corrections. However, these studies are based on the probe limit which neglects the back reactions of matter fields on the spacetime metric. Holographic superconductors in the presence of Born-Infeld (BI) electrodynamics in Gauss-Bonnet (GB) background with backreaction has been investigated analytically using Sturm-Liouville (SL) eigenvalue method in our first work [35]. This study incorporates the non-linear effects on holographic superconductor since we consider non-linear electrodynamics. The physical motivation of looking at the leading order corrections coming from the Born-Infeld coupling parameter is to investigate the effects due to higher derivative corrections of gauge fields on the order parameter condensation. The theory is important in its own right as it removes the divergence in the self energy of point charged particles and also enjoys electromagnetic duality [58]-[60]. The Gauss-Bonnet (GB) gravity [151]-[152] has attracted a lot of attention among gravity theories with higher curvature corrections. The Mermin-Wagner theorem suggests that the phase transition may be affected by higher curvature corrections. Investigations in this direction led to the introduction of a new analytic method in [74], the so-called matching method, which is based on the solution to the field equations near the horizon and near the asymptotic region and then matching the two solutions at some intermediate point. In this chapter, we have employed the

SL eigenvalue method to estimate the critical temperature and the condensation operator value. This section is based on our work [35].

The action for the formation of scalar hair on an electrically charged black hole in d-dimensional anti-de Sitter spacetime reads

$$S = \int d^d x \frac{\sqrt{-g}}{2\kappa^2} \left(R - 2\Lambda + \frac{\alpha}{2}(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}) + 2\kappa^2 \mathcal{L}_{matter} \right) \tag{3.70}$$

where $\Lambda = -(d-1)(d-2)/(2L^2)$ is the cosmological constant, $\kappa^2 = 8\pi G_d$ is the d-dimensional Newton's gravitational constant and α is the Gauss-Bonnet coupling parameter. The matter Lagrangian density is denoted by $\mathcal{L}_{matter}$ which is

$$\mathcal{L}_{matter} = \mathcal{L}_{BI} - (D_\mu \psi)^* D^\mu \psi - m^2 \psi^* \psi \tag{3.71}$$

where

$$\mathcal{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{b}{2}F^{\alpha\beta}F_{\alpha\beta} - \frac{b^2}{16}(G^{\alpha\beta}F_{\alpha\beta})^2}\right) . \tag{3.72}$$

$\mathcal{L}_{BI}$ denotes the Lagrangian for Born-Infeld (BI) electrodynamics [59, 60] where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; $(\mu, \nu = 0, 1, 2, 3, 4)$ is the field strength tensor, $G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ is dual field tensor, A_μ represents the gauge field. $D_\mu\psi = \partial_\mu\psi - iqA_\mu\psi$ is the covariant derivative and ψ is the scalar field.
The ansatz for the gauge field and the scalar field is now chosen to be [34]

$$A = \phi(r)dt \quad , \quad \psi = \psi(r) \tag{3.73}$$

for studying the critical phenomena of holographic superconductors. The above ansatz implies that the black hole possesses only electric charge. Note that the last term in $\mathcal{L}_{BI}$ vanishes since $\vec{A} = 0$. So the effective BI Lagrangian reads for investigation of the critical temperature and condensation

$$\mathcal{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{b}{2}F^{\alpha\beta}F_{\alpha\beta}}\right) \tag{3.74}$$

which is known as Born electrodynamics [58].
We now assume that the plane-symmetric black hole metric with back reaction can be written in the form

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^i dx^j \tag{3.75}$$

where $h_{ij}dx^i dx^j$ denotes the line element of a $(d-2)$-dimensional hypersurface with zero curvature. The Hawking temperature of this black hole, which is interpreted as the temperature of the conformal field theory on the boundary, is given by

$$T_H = \frac{f'(r_+)e^{-\chi(r_+)/2}}{4\pi} \tag{3.76}$$

where r_+ is the radius of the horizon of the black hole.

The equations of motion for the metric and matter fields calculated with this ansatz read

$$\left\{1-\frac{(d-4)(d-3)\alpha f(r)}{r^2}\right\}f'(r)+\frac{(d-3)f(r)}{r}-\frac{(d-1)r}{L^2}-\frac{(d-5)(d-4)(d-3)}{2r^3}\alpha f^2(r)$$
$$+\frac{2\kappa^2 r}{d-2}\left[f(r)\psi'(r)^2+\frac{q^2\phi^2(r)\psi^2(r)e^{\chi(r)}}{f(r)}+m^2\psi^2(r)+\frac{1}{b}\left((1-b\phi'(r)^2)^{-\frac{1}{2}}-1\right)\right]=0 \tag{3.77}$$

$$\left\{1-\frac{(d-4)(d-3)\alpha f(r)}{r^2}\right\}\chi'(r)+\frac{4\kappa^2 r}{d-2}\left(\psi'(r)^2+\frac{q^2\phi^2(r)\psi^2(r)e^{\chi(r)}}{f(r)^2}\right)=0 \tag{3.78}$$

$$\phi''(r)+\left(\frac{d-2}{r}+\frac{\chi'(r)}{2}\right)\phi'(r)-\frac{d-2}{r}be^{\chi(r)}\phi'(r)^3-\frac{2q^2\phi(r)\psi^2(r)}{f(r)}(1-be^{\chi(r)}\phi'(r)^2)^{\frac{3}{2}}=0 \tag{3.79}$$

$$\psi''(r)+\left(\frac{d-2}{r}-\frac{\chi'(r)}{2}+\frac{f'(r)}{f(r)}\right)\psi'(r)+\left(\frac{q^2\phi^2(r)e^{\chi(r)}}{f(r)^2}-\frac{m^2}{f(r)}\right)\psi(r)=0 \tag{3.80}$$

where prime denotes derivative with respect to r. The fact that $\kappa \neq 0$ takes into account the backreaction of the spacetime. Without any loss of generality, this limit also enables one to choose $q=1$ since the rescalings $\psi \to \psi/q$, $\phi \to \phi/q$ and $\kappa^2 \to q^2\kappa^2$ can be performed [134].
We now proceed to solve the non-linear equations (3.77)-(3.80). In order to do this we need to fix the boundary conditions for $\phi(r)$ and $\psi(r)$ at the black hole horizon $r=r_+$ (where $f(r=r_+)=0$ with $e^{-\chi(r=r_+)}$ finite) and at the spatial infinity ($r\to\infty$). For the matter fields to be regular, we require $\phi(r_+)=0$ and $\psi(r_+)$ to be finite at the horizon.
Near the boundary of the bulk, we can set $e^{-\chi(r\to\infty)}\to 1$, so that the spacetime becomes a Reissner-Nordström-anti-de Sitter black hole. The matter fields there obey [34]

$$\phi(r) = \mu-\frac{\rho}{r^{d-3}} \tag{3.81}$$
$$\psi(r) = \frac{\psi_-}{r^{\Delta_-}}+\frac{\psi_+}{r^{\Delta_+}} \tag{3.82}$$

where

$$\Delta_\pm = \frac{(d-1)\pm\sqrt{(d-1)^2+4m^2L^2}}{2}\,. \tag{3.83}$$

The parameters μ and ρ are dual to the chemical potential and charge density of the conformal field theory on the boundary. We choose $\psi_-=0$, so that ψ_+ is dual to the expectation value of the condensation operator J at the boundary.

Under the change of coordinates $z = \frac{r_+}{r}$, the field equations (3.77)-(3.80) become

$$\left(1 - \frac{(d-4)(d-3)\alpha z^2 f(z)}{r_+^2}\right) f'(z) - \frac{(d-3)f(z)}{z} + \frac{(d-1)r_+^2}{L^2 z^3} + \frac{(d-5)(d-4)(d-3)z}{2r_+^2}\alpha f^2(z)$$
$$-\frac{2\kappa^2 r_+^2}{(d-2)z^3}\left[\frac{z^4}{r_+^2} f(z)\psi'(z)^2 + \frac{\phi^2(z)\psi^2(z)e^{\chi(z)}}{f(z)} + m^2\psi^2(z) + \frac{1}{b}\left((1 - \frac{bz^4}{r_+^2}\phi'(z)^2)^{-\frac{1}{2}} - 1\right)\right] = 0 \tag{3.84}$$

$$\left(1 - \frac{(d-4)(d-3)\alpha z^2 f(z)}{r_+^2}\right)\chi'(z) - \frac{4\kappa^2 r_+^2}{(d-2)z^3}\left(\frac{z^4}{r_+^2}\psi'(z)^2 + \frac{\phi^2(z)\psi^2(z)e^{\chi(z)}}{f(z)^2}\right) = 0 \tag{3.85}$$

$$\phi''(z) + \left(\frac{\chi'(z)}{2} - \frac{d-4}{z}\right)\phi'(z) + \frac{d-2}{r_+^2}be^{\chi(z)}\phi'(z)^3 z^3 - \frac{2r_+^2\phi(z)\psi^2(z)}{f(z)z^4}\left(1 - \frac{bz^4e^{\chi(z)}}{r_+^2}\phi'(z)^2\right)^{\frac{3}{2}} = 0 \tag{3.86}$$

$$\psi''(z) + \left(\frac{f'(z)}{f(z)} - \frac{d-4}{z} - \frac{\chi'(z)}{2}\right)\psi'(z) + \frac{r_+^2}{z^4}\left(\frac{\phi^2(z)e^{\chi(z)}}{f(z)^2} - \frac{m^2}{f(z)}\right)\psi(z) = 0 \tag{3.87}$$

where prime now denotes derivative with respect to z. These equations are to be solved in the interval $(0,1)$, where $z=1$ is the horizon and $z=0$ is the boundary. The boundary condition $\phi(r_+) = 0$ now translates to $\phi(z=1) = 0$.

3.4.1 The critical temperature T_c

With the basic formalism in place, in this section we shall proceed to investigate the relation between the critical temperature and the charge density. To begin with, we first need to obtain a solution of the above equations.
At the critical temperature T_c, $\psi = 0$, hence eq.(3.85) reduces to

$$\chi'(z) = 0. \tag{3.88}$$

Near the boundary of the bulk, we can set $e^{-\chi(r\to\infty)} \to 1$, that is $\chi(r\to\infty) = 0$ which in turn implies $\chi(z) = 0$ from eq.(3.88). The field equation (3.86) therefore reduces to

$$\phi''(z) - \frac{d-4}{z}\phi'(z) + \frac{(d-2)bz^3}{r_{+(c)}^2}\phi'(z)^3 = 0. \tag{3.89}$$

To solve this non-linear differential equation, we take recourse to a perturbative technique developed in [129]. When $b=0$, the above equation becomes

$$\phi''(z) - \frac{d-4}{z}\phi'(z) = 0. \tag{3.90}$$

Using the asymptotic behaviour of $\phi(z)$ (eq.(3.81)), the solution of eq.(3.90) yields

$$\phi(z)|_{b=0} = \lambda r_{+(c)}(1 - z^{d-3}) \tag{3.91}$$

where

$$\lambda = \frac{\rho}{r_{+(c)}^{d-2}} \ . \tag{3.92}$$

To solve eq.(3.89), we put the solution for $\phi(z)$ with $b = 0$ (i.e. $\phi(z)|_{b=0}$) in the non-linear term of eq.(3.89). This leads to

$$\phi''(z) - \frac{d-4}{z}\phi'(z) - b\lambda^3 r_{+(c)}(d-2)(d-3)^3 z^{3(d-3)} = 0. \tag{3.93}$$

Using the asymptotic boundary condition (3.81), the solution of the above equation upto first order in the Born-Infeld parameter b reads

$$\phi(z) = \lambda r_{+(c)} \left\{ (1 - z^{d-3}) - \frac{b(\lambda^2|_{b=0})(d-3)^3}{2(3d-7)}(1 - z^{3d-7}) \right\} \tag{3.94}$$

where we have used the fact that $b\lambda^2 = b(\lambda^2|_{b=0}) + \mathcal{O}(b^2)$ [129], $\lambda^2|_{b=0}$ being the value of λ^2 for $b = 0$. It is reassuring to note that the above result agrees with solution of $\phi(z)$ obtained in [129] for $d = 4$.

Backreaction effect in Einstein gravity

For Einstein gravity $\alpha = 0$, eq.(3.84) at $T = T_c$ then becomes

$$f'(z) - \frac{d-3}{z}f(z) + \frac{(d-1)r_{+(c)}^2}{L^2 z^3} - \frac{2\kappa^2 r_{+(c)}^2}{b(d-2)z^3}\left([1 - \frac{bz^4}{r_{+(c)}^2}\phi'(z)^2]^{-\frac{1}{2}} - 1 \right) = 0. \tag{3.95}$$

Dropping terms of the order of $b\kappa^2$, eq.(3.95) reduces to

$$f'(z) - \frac{d-3}{z}f(z) + \frac{(d-1)r_{+(c)}^2}{L^2 z^3} - \frac{\kappa^2 z}{d-2}\phi'(z)^2 = 0 \ . \tag{3.96}$$

Substituting $\phi(z)|_{b=0}$ (eq.(3.91)) in the above equation leads to

$$f'(z) - \frac{d-3}{z}f(z) + \frac{(d-1)r_{+(c)}^2}{L^2 z^3} - \frac{\kappa^2\lambda^2 r_{+(c)}^2(d-3)^2}{d-2}z^{2d-7} = 0. \tag{3.97}$$

The solution of the metric from eq.(3.97) subject to the condition $f(z = 1) = 0$ reads

$$f(z) \ = \ r_{+(c)}^2 \left\{ \frac{1}{L^2 z^2} - \left(\frac{1}{L^2} + \frac{d-3}{d-2}\kappa^2\lambda^2 \right) z^{d-3} + \frac{d-3}{d-2}\kappa^2\lambda^2 z^{2(d-3)} \right\}. \tag{3.98}$$

In the rest of the analysis we shall set $L = 1$. Eq.(3.98) therefore reads

$$f(z) = \frac{r_{+(c)}^2}{z^2} g_0(z) \tag{3.99}$$

where

$$g_0(z) = 1 - \left(1 + \frac{d-3}{d-2}\kappa^2\lambda^2\right) z^{d-1} + \frac{d-3}{d-2}\kappa^2\lambda^2 z^{2(d-2)}. \tag{3.100}$$

Now we find that as $T \to T_c$, eq.(3.87) for the field ψ reads

$$\psi''(z) + \left(\frac{g_0'(z)}{g_0(z)} - \frac{d-2}{z}\right)\psi'(z) + \left(\frac{\phi^2(z)}{g_0^2(z) r_{+(c)}^2} - \frac{m^2}{g_0(z) z^2}\right)\psi(z) = 0 \tag{3.101}$$

where $\phi(z)$ now corresponds to the solution in eq.(3.94). In the above equation, we shall also consider the fact that $\kappa_i^2 \lambda^2 = \kappa_i^2(\lambda^2|_{\kappa_{i-1}}) + \mathcal{O}(\kappa^4)$ which in turn implies that $g_0(z)$ takes the following form

$$g_0(z) = 1 - \left(1 + \frac{d-3}{d-2}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) z^{d-1} + \frac{d-3}{d-2}\kappa_i^2(\lambda^2|_{\kappa_{i-1}}) z^{2(d-2)}. \tag{3.102}$$

Near the boundary, we define [75]

$$\psi(z) = \frac{\langle J \rangle}{r_{+(c)}^{\Delta_+}} z^{\Delta_+} F(z) \tag{3.103}$$

where $F(0) = 1$ and J is the condensation operator. Substituting this form of $\psi(z)$ in eq.(3.87), we obtain

$$\begin{aligned} F''(z) &+ \left\{\frac{2\Delta_+ - d + 2}{z} + \frac{g_0'(z)}{g_0(z)}\right\} F'(z) \\ &+ \left\{\frac{\Delta_+(\Delta_+ - 1)}{z^2} + \left(\frac{g_0'(z)}{g_0(z)} - \frac{d-2}{z}\right)\frac{\Delta_+}{z} - \frac{m^2}{g_0(z) z^2}\right\} F(z) \\ &+ \frac{\lambda^2}{g_0^2}\left\{(1 - z^{d-3})^2 - \frac{b(\lambda^2|_{b=0})(d-3)^3}{3d-7}(1 - z^{d-3})(1 - z^{3d-7})\right\} F(z) = 0 \end{aligned} \tag{3.104}$$

to be solved subject to the boundary condition $F'(0) = 0$.
It is now simple to see that the above equation can be written in the Sturm-Liouville form

$$\frac{d}{dz}\{p(z)F'(z)\} + q(z)F(z) + \lambda^2 r(z) F(z) = 0 \tag{3.105}$$

with

$$\begin{aligned} p(z) &= z^{2\Delta_+ - d + 2} g_0(z) \\ q(z) &= z^{2\Delta_+ - d + 2} g_0(z) \left\{\frac{\Delta_+(\Delta_+ - 1)}{z^2} + \left(\frac{g_0'(z)}{g_0(z)} - \frac{d-2}{z}\right)\frac{\Delta_+}{z} - \frac{m^2}{g_0(z) z^2}\right\} \\ r(z) &= \frac{z^{2\Delta_+ - d + 2}}{g_0(z)}\left\{(1 - z^{d-3})^2 - \frac{b(\lambda^2|_{b=0})(d-3)^3}{3d-7}(1 - z^{d-3})(1 - z^{3d-7})\right\}. \end{aligned} \tag{3.106}$$

The above identification enables us to write down an equation for the eigenvalue λ^2 which minimizes the expression

$$\lambda^2 = \frac{\int_0^1 dz \, \{p(z)[F'(z)]^2 + q(z)[F(z)]^2\}}{\int_0^1 dz \, r(z)[F(z)]^2} . \tag{3.107}$$

We shall now use the following trial function for the estimation of λ^2

$$F = F_{\tilde{\alpha}}(z) \equiv 1 - \tilde{\alpha} z^2 . \tag{3.108}$$

Note that F satisfies the conditions $F(0) = 1$ and $F'(0) = 0$.
Using eq.(3.76) and eq.(s)(3.99, 3.100), we get the relation between the critical temperature and the charge density

$$T_c = \frac{1}{4\pi}\left[(d-1) - \frac{(d-3)^2}{(d-2)}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right]\left(\frac{\rho}{\lambda}\right)^{\frac{1}{d-2}} . \tag{3.109}$$

The above result holds for a d-dimensional holographic superconductor and is one of the main results obtained in this paper [35]. It is to be noted that the effect of the BI coupling parameter b in the critical temperature T_c comes through the eigenvalue λ. In the rest of our analysis, we shall set $d = 5$ and $m^2 = -3$. The choice for m^2 yields $\Delta_+ = 3$ from eq.(3.83). Eq.(s)(3.109, 3.106) therefore becomes

$$T_c = \frac{1}{\pi}\left[1 - \frac{1}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right]\left(\frac{\rho}{\lambda}\right)^{\frac{1}{3}} \tag{3.110}$$

$$\begin{aligned} p(z) &= z^3\left\{1 - z^4\left(1 + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})z^6\right\} \\ q(z) &= -9z^5\left(1 + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) + 10\kappa_i^2(\lambda^2|_{\kappa_{i-1}})z^7 \\ r(z) &= \frac{z^3\left\{(1-z^2)^2 - b(\lambda^2|_{b=0})(1-z^2)(1-z^8)\right\}}{1 - z^4\left(1 + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})z^6} . \end{aligned} \tag{3.111}$$

With the backreaction parameter $\kappa = 0$ and Born-Infeld parameter $b = 0$, the trial function (3.108) and eq.(3.111) leads to

$$\lambda_{\tilde{\alpha}}^2 = \frac{2(18 - 27\tilde{\alpha} + 14\tilde{\alpha}^2)}{6(3 - 4\ln 2) + 16(2 - 3\ln 2)\tilde{\alpha} + (17 - 24\ln 2)\tilde{\alpha}^2} . \tag{3.112}$$

This expression attains its minimum at $\tilde{\alpha} \approx 0.7218$. The critical temperature can now be computed from eq.(3.110) and reads

$$T_c = \frac{1}{\pi(\lambda|_{\tilde{\alpha}=0.7218})^{1/3}}\rho^{1/3} \approx 0.1962\rho^{1/3} \tag{3.113}$$

which is in very good agreement with the numerical value $T_c = 0.1980\rho^{1/3}$ [72].

Now in order to include the effect of the Born-Infeld parameter b, we set $b = 0.01$ and rerun the above analysis to get the value of λ^2 for $b = 0.01$

$$\lambda_{\tilde{\alpha}}^2 = \frac{1.500 - 2.250\tilde{\alpha} + 1.6667\tilde{\alpha}^2}{0.0371037 - 0.0316927\tilde{\alpha} + 0.00845841\tilde{\alpha}^2} \tag{3.114}$$

Table 3.1: The critical temperature for backreaction parameter $\kappa = 0$

b	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\|_{SL}$	$(T_c/\rho^{1/3})\|_{numerical}$
0.0	0.7218	18.23	0.1962	0.1980
0.01	0.7540	25.91	0.1850	0.1910
0.02	0.8201	44.08	0.1694	0.1851

which attains its minimum at $\tilde{\alpha} \approx 0.7540$. The critical temperature therefore reads

$$T_c = \frac{1}{\pi \lambda^{1/3}_{\tilde{\alpha}=0.7540}} \rho^{1/3} \approx 0.1850 \rho^{1/3} \tag{3.115}$$

which is in very good agreement with the numerical value of T_c which is $T_c = 0.1910\rho^{1/3}$ [144].
Setting $b = 0.02$ yields

$$\lambda^2_{\tilde{\alpha}} = \frac{1.500 - 2.250\tilde{\alpha} + 1.6667\tilde{\alpha}^2}{0.0173545 - 0.0104244\tilde{\alpha} + 0.00173068\tilde{\alpha}^2} \tag{3.116}$$

which attains its minimum at $\tilde{\alpha} \approx 0.8201$. Hence the critical temperature reads

$$T_c = \frac{1}{\pi \lambda^{1/3}_{\tilde{\alpha}=0.8201}} \rho^{1/3} \approx 0.1694 \rho^{1/3} \tag{3.117}$$

which is once again in good agreement with the numerical value $T_c = 0.1851\rho^{1/3}$ [144]. A comparison of the analytical and numerical results for the critical temperature and the charge density in Einstein gravity with backreaction parameter $\kappa = 0$ is presented in Table 3.1.

Now we shall proceed to include the effect of backreaction ($\kappa \neq 0$) in the above analysis. To do this, we shall increase the value of κ in steps of 0.05. To begin with, we set $b = 0$ and the backreaction parameter $\kappa = 0.05$. Rerunning the above procedure using eq.(s)(3.108, 3.111) leads to

$$\lambda^2_{\tilde{\alpha}} = \frac{1.48861 - 2.22721\tilde{\alpha} + 1.15401\tilde{\alpha}^2}{0.057139 - 0.0533188\tilde{\alpha} + 0.0153081\tilde{\alpha}^2} \ . \tag{3.118}$$

This attains its minimum at $\tilde{\alpha} \approx 0.7195$. The critical temperature therefore reads

$$T_c = \frac{1}{\pi} \frac{(1 - \frac{1}{3}\kappa^2 \lambda^2_{\kappa=0.0})}{\lambda^{1/3}_{\tilde{\alpha}=0.7195}} \rho^{1/3} \approx 0.1934 \rho^{1/3} \tag{3.119}$$

which is in good agreement with the numerical value $T_c = 0.1953\rho^{1/3}$ [147]. We repeat our calculations of T_c for the same value of κ but with different values of b. The critical temperature reads $T_c = 0.1825\rho^{1/3}$ and $T_c = 0.1672\rho^{1/3}$ for $b = 0.01,\ 0.02$ respectively. Next we repeat the same analysis for $\kappa = 0.10$ and $\kappa = 0.15$. Figure 3.2 shows the plot of T_c vs. ρ for Einstein holographic superconductors for different choice of parameters $\kappa,\ b$. The plots clearly show that the condensation becomes harder to form as the values of the backreaction parameter κ and BI coupling parameter b are increased.
In Table 3.2, we present our analytical values obtained by the SL eigenvalue approach for different sets of values of b and κ. In Fig. 3.2, we show the effect of backreaction as well as BI coupling parameters on the critical temperature (T_c).

Table 3.2: The analytical results for the critical temperature with backreaction and Born-Infeld parameter in Einstein gravity

κ	b	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\vert_{SL}$
	0.0	0.7195	18.11	0.1934
0.05	0.01	0.7525	25.68	0.1825
	0.02	0.8203	43.46	0.1672
	0.0	0.7122	17.75	0.1852
0.10	0.01	0.7455	25.02	0.1751
	0.02	0.8148	41.73	0.1608
	0.0	0.6995	17.16	0.1718
0.15	0.01	0.7345	23.99	0.1634
	0.02	0.8024	39.07	0.1506

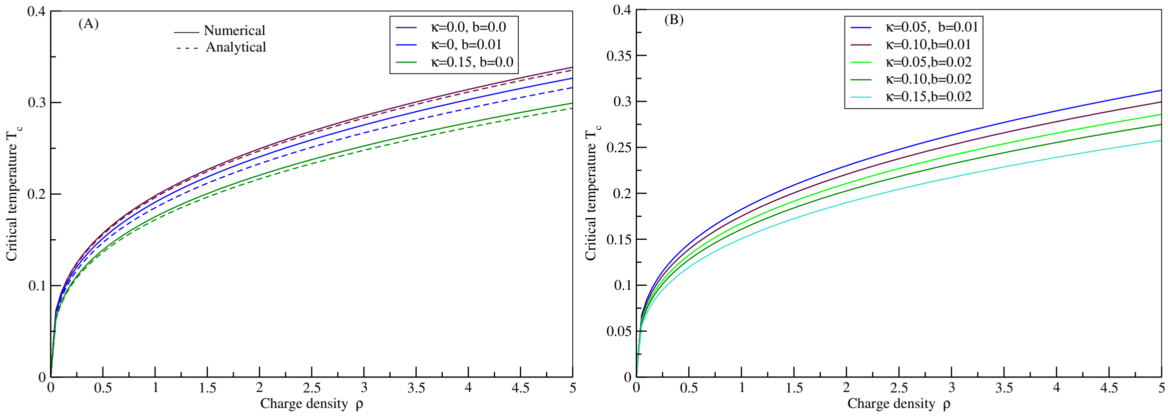

Figure 3.2: T_c vs. ρ plot for Einstein holographic superconductors for different choice of parameters backreaction parameter κ and BI parameter b. (A) The continuous curves correspond to numerical values whereas the dotted curves correspond to analytic values for $(\kappa = 0.0,\ b = 0.0)$, $(\kappa = 0,\ b = 0.01)$, $(\kappa = 0.15,\ b = 0.0)$. (B) All curves correspond to analytic values for $(\kappa = 0.05,\ b = 0.01)$, $(\kappa = 0.10,\ b = 0.01)$, $(\kappa = 0.05,\ b = 0.02)$, $(\kappa = 0.10,\ b = 0.02)$, $(\kappa = 0.15,\ b = 0.02)$ (upper to lower).

Backreaction effect in Gauss-Bonnet gravity

In this subsection, we study the relation between the critical temperature and the charge density taking into account the effect of the Gauss-Bonnet coupling parameter α. It is to be noted that since κ and b are very small, hence we shall neglect $\mathcal{O}(b\kappa^2)$ and higher order terms in our analysis.
In this case, using eq.(3.84), eq.(3.88) (with $\alpha \neq 0$ and $L = 1$) reduces to

$$\left(1 - \frac{(d-4)(d-3)\alpha z^2}{r^2_{+(c)}} f(z)\right) f'(z) \quad - \quad \frac{d-3}{z} f(z) + \frac{(d-1) r^2_{+(c)}}{z^3} - \frac{\kappa^2 z}{d-2} \phi'(z)^2$$
$$+ \frac{(d-5)(d-4)(d-3)\alpha z}{2 r^2_{+(c)}} f^2(z) \quad = \quad 0. \tag{3.120}$$

Since we are not concerned with terms of the order of $b\kappa^2$, we substitute $\phi(z)|_{b=0}$

(eq.(3.91)) in eq.(3.120). The metric equation then becomes

$$
\begin{aligned}
f'(z) &- \frac{d-3}{z}f(z) + \frac{(d-1)r_{+(c)}^2}{L^2 z^3} - \frac{\kappa^2\lambda^2 r_{+(c)}^2 (d-3)^2}{d-2} z^{2d-7} \\
&= \frac{(d-4)(d-3)\alpha}{r_{+(c)}^2}\left[z^2 f(z) f'(z) - \frac{(d-5)}{2} z f^2(z)\right] . \qquad (3.121)
\end{aligned}
$$

To solve this non-linear equation, we once again employ the perturbative technique. First we consider $\alpha = 0$ for which we know the solution, namely, $f(z)|_{\alpha=0} = \frac{r_{+(c)}^2}{z^2} g_0(z)$. To solve for $\alpha \neq 0$, we substitute $f(z)|_{\alpha=0}$ and $f'(z)|_{\alpha=0}$ in the right hand side of eq.(3.121). The solution of the above equation upto first order in the Gauss-Bonnet coupling parameter α therefore reads

$$
f(z) = \frac{r_{+(c)}^2}{z^2}\left\{g_0(z) + (d-4)(d-3)\alpha g_1(z)\right\} \qquad (3.122)
$$

where

$$
\begin{aligned}
g_1(z) &= \frac{2}{d-1} - z^{d-1} - (d-5) z^{d-1}\log z + \frac{d-3}{d-1} z^{2(d-1)} \\
&+ \left\{\frac{2(d-4)}{d-2} z^{2(d-2)} - \frac{3(d-3)^2}{2(d-2)^2} z^{3d-5} + \frac{2(d-3)^2}{(d-2)(d-1)} z^{2(d-1)}\right\} \kappa_i^2(\lambda^2|_{\kappa_{i-1}}) \\
&+ z^{d-1}\left\{\frac{77 - 95d + 39d^2 - 5d^3}{2(d-1)(d-2)^2} - \frac{(d-3)(d-5)}{d-2}\log z\right\} \kappa_i^2(\lambda^2|_{\kappa_{i-1}}) \\
&- \frac{d-5}{2}\left\{-\frac{1}{d-1} + \frac{z^{2(d-1)}}{d-1} - 2z^{d-1}\log z\right\} \\
&- \frac{d-5}{2}\left\{\frac{2(d-3)}{(d-1)(d-2)} z^{2(d-1)} - \frac{(d-3)}{(d-2)^2} z^{3d-5} + \frac{2}{(d-2)} z^{2(d-2)}\right\} \kappa_i^2(\lambda^2|_{\kappa_{i-1}}) \\
&+ \frac{d-5}{2} z^{d-1}\left\{\frac{2(d-3)}{(d-1)(d-2)} + \frac{2}{d-2} - \frac{d-3}{(d-2)^2} + \frac{2(d-3)}{d-2}\log z\right\} \kappa_i^2(\lambda^2|_{\kappa_{i-1}}) \\
\Rightarrow g_1(z) &= \frac{1}{2}\left[1 - 2z^{d-1} + z^{2(d-1)} - \frac{2(d-3)}{d-2}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\left\{z^{d-1} - z^{2(d-1)} - z^{2(d-2)} + z^{3d-5}\right\}\right] .
\end{aligned}
$$

(3.123)

Once again substituting the form $\psi(z)$ near the boundary (defined in eq.(3.103)) in eq.(3.101), we obtain

$$
\begin{aligned}
F''(z) &+ \left\{\frac{2\Delta_+ - d + 2}{z} + \frac{g_0'(z) + (d-4)(d-3)\alpha g_1'(z)}{g_0(z) + (d-4)(d-3)\alpha g_1(z)}\right\} F'(z) \\
&+ \left\{\frac{\Delta_+(\Delta_+ - 1)}{z^2} + \left(\frac{g_0'(z) + (d-4)(d-3)\alpha g_1'(z)}{g_0(z) + (d-4)(d-3)\alpha g_1(z)} - \frac{d-2}{z}\right)\frac{\Delta_+}{z}\right. \\
&\left. - \frac{m^2}{(g_0(z) + (d-4)(d-3)\alpha g_1(z)) z^2}\right\} F(z) \\
&+ \frac{\phi^2(z)|_{b=0}}{r_{+(c)}^2 \left(g_0(z) + (d-4)(d-3)\alpha g_1(z)\right)^2} F(z) = 0 \qquad (3.124)
\end{aligned}
$$

to be solved subject to the boundary condition $F'(0) = 0$.
The above equation can once again be put in the Sturm-Liouville form (3.105) with

$$\begin{aligned}
p(z) &= z^{2\Delta_+ - d+2}\left\{g_0(z) + (d-4)(d-3)\alpha g_1(z)\right\} \\
q(z) &= z^{2\Delta_+ - d+2}\left\{g_0(z) + (d-4)(d-3)\alpha g_1(z)\right\}\left\{\frac{\Delta_+(\Delta_+ - d+1)}{z^2}\right. \\
&\quad + \left.\left(\frac{g_0'(z) + (d-4)(d-3)\alpha g_1'(z)}{g_0(z) + (d-4)(d-3)\alpha g_1(z)}\right)\frac{\Delta_+}{z} - \frac{m^2}{(g_0(z) + (d-4)(d-3)\alpha g_1(z))\, z^2}\right\} \\
r(z) &= \frac{z^{2\Delta_+ - d+2}}{(g_0(z) + (d-4)(d-3)\alpha g_1(z))}\left\{(1 - z^{d-3})^2 - \frac{\lambda^2\,|_{b=0}\, b(d-3)^3}{3d-7}(1 - z^{d-3})(1 - z^{3d-7})\right\}.
\end{aligned} \tag{3.125}$$

With the above identification, we can once again proceed to find the minimum value of the eigenvalue λ^2 as in the earlier section.
Once again using eq.(3.76) and eq.(s)(3.122, 3.123), we get the relation between the critical temperature and the charge density.
It is to be noted that the expression for the critical temperature in GB gravity is identical to the corresponding expression in Einstein gravity (3.109). This is because $g_1'(z)$ vanishes at $z = 1$. However, their numerical values will be different since in GB gravity, the eigenvalues λ will be affected by the GB coupling parameter α.

Setting $d = 5$ and $m^2 = -3$, eq.(s)(3.109, 3.125) become

$$T_c = \frac{1}{\pi}\left[1 - \frac{1}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right]\left(\frac{\rho}{\lambda}\right)^{\frac{1}{3}} \tag{3.126}$$

$$\begin{aligned}
p(z) &= z^3\left\{1 - z^4\left(1 + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})z^6\right\} \\
&\quad + 2\alpha z^3\left\{\frac{1}{2}(1 + z^8) - z^4 - \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\left(z^4 - z^6 - z^8 + z^{10}\right)\right\} \\
q(z) &= -9z^5\left(1 + \frac{2}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\right) + 10\kappa_i^2(\lambda^2|_{\kappa_{i-1}})z^7 \\
&\quad + \alpha\left\{21z^9 - 18z^5 - 3z - 4\kappa_i^2(\lambda^2|_{\kappa_{i-1}})\left(3z^5 - 5z^7 - 7z^9 + 9z^{11}\right)\right\} \\
r(z) &= \frac{z^3\left\{(1 - z^2)^2 - b(\lambda^2|_{b=0})(1 - z^2)(1 - z^8)\right\}}{1 - z^4\left(1 + \frac{2}{3}\kappa_i^2\lambda^2|_{\kappa_{i-1}}\right) + \frac{2}{3}\kappa_i^2\lambda^2|_{\kappa_{i-1}}z^6 + 2\alpha\left\{\frac{1}{2}(1 + z^8) - z^4 - \frac{2}{3}\kappa^2\lambda^2\left(z^4 - z^6 - z^8 + z^{10}\right)\right\}}.
\end{aligned} \tag{3.127}$$

To estimate λ^2, we first set $\alpha = -0.1$, $\kappa = 0$, $b = 0$ and once again use the trial function (3.108) to obtain

$$\lambda_{\tilde{\alpha}}^2 = \frac{1.26 - 2.00\tilde{\alpha} + 1.07143\tilde{\alpha}^2}{0.0613835 - 0.0565523\tilde{\alpha} + 0.0160771\tilde{\alpha}^2} \tag{3.128}$$

which attains its minimum at $\tilde{\alpha} \approx 0.7305$. The critical temperature therefore reads

$$T_c = \frac{1}{\pi\lambda_{\tilde{\alpha}=0.7305}^{1/3}}\rho^{1/3} \approx 0.208\rho^{1/3} \tag{3.129}$$

Table 3.3: The critical temperature for Gauss-Bonnet parameter $\alpha = -0.1$ and Born-Infeld parameter $b = 0$

κ	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\|_{SL}$	$(T_c/\rho^{1/3})\|_{numerical}$
0.0	0.7305	12.940	0.2078	0.2090
0.01	0.7345	12.937	0.2077	0.2089
0.02	0.7302	12.930	0.2074	0.2087

Table 3.4: The critical temperature for $\alpha = 0.0001$ and $\kappa = 0$

b	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\|_{SL}$	$(T_c/\rho^{1/3})\|_{numerical}$
0.0	0.7218	18.2358	0.1962	0.1962
0.01	0.7565	25.8432	0.1851	0.1910
0.02	0.8211	44.105	0.1693	0.1851

which is in very good agreement with the exact $T_c = 0.209\rho^{1/3}$ [150].
Next we include effect of backreaction. We calculate λ^2 for $\kappa = 0.01,\ b = 0$ which attains its minimum at $\tilde{\alpha} \approx 0.7345$. The critical temperature therefore reads

$$T_c = \frac{1}{\pi}\frac{\left(1 - \frac{1}{3}\kappa^2\lambda^2_{\kappa=0.0}\right)}{\lambda^{1/3}_{\tilde{\alpha}=0.7345}}\rho^{1/3} \approx 0.2077\rho^{1/3} \tag{3.130}$$

which is in very good agreement with the exact $T_c = 0.2089\rho^{1/3}$ [149].
In the Tables below, we present the analytical results obtained by the SL approach for different sets of values of α, κ and b (Tables 3.3,3.4).
In Fig. 3.3, the plot of T_c vs. ρ is shown for holographic superconductors in the framework of Gauss-Bonnet gravity for different choice of parameters κ, b. The plots clearly show that the condensation becomes harder to form as the values of the backreaction parameter κ, BI coupling parameter b and the GB parameter α are increased (Tables 3.5, 3.6).

Table 3.5: The analytical results for the critical temperature with backreaction and Born-Infeld parameter in Gauss-Bonnet gravity ($\alpha = 0.0001$)

κ	b	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\|_{SL}$
0.05	0.0	0.7205	18.12	0.1934
	0.01	0.7505	25.75	0.1824
	0.02	0.8275	42.93	0.1675
0.10	0.0	0.7125	17.75	0.1852
	0.01	0.7454	25.04	0.1751
	0.02	0.8135	41.70	0.1608
0.15	0.0	0.7011	17.16	0.1718
	0.01	0.7318	24.00	0.1633
	0.02	0.8025	39.10	0.1505

Table 3.6: The analytical results for the critical temperature with backreaction and Born-Infeld parameter in Gauss-Bonnet gravity ($\alpha = 0.1$)

κ	b	$\tilde{\alpha}$	λ^2_{SL}	$(T_c/\rho^{1/3})\|_{SL}$
	0.0	0.7080	24.18	0.1872
0.0	0.01	0.7665	39.96	0.1722
	0.02	0.9375	103.31	0.1470
	0.0	0.7053	23.96	0.1837
0.05	0.01	0.7645	39.42	0.1691
	0.02	0.9345	100.189	0.1448
	0.0	0.6935	23.30	0.1733
0.10	0.01	0.7505	37.88	0.1602
	0.02	0.9200	91.67	0.1382
	0.0	0.6705	22.24	0.1566
0.15	0.01	0.7390	35.50	0.1463
	0.02	0.9010	79.92	0.1278

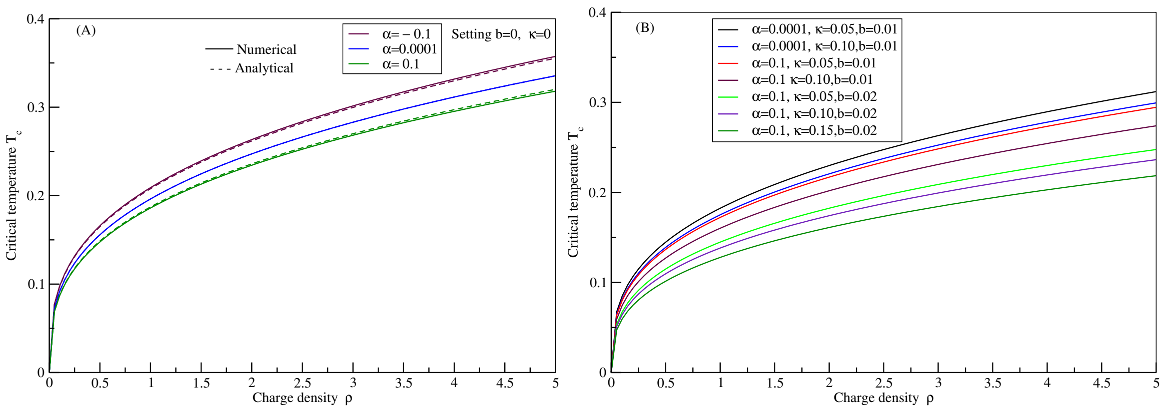

Figure 3.3: (A) T_c vs. ρ plot for Gauss-Bonnet holographic superconductors for different choice of GB parameters α (three sets) with the same value of $\kappa = 0.0,\ b = 0$. The continuous curves correspond to numerical values whereas the dotted curves correspond to analytic values for $\alpha = -0.1,\ \alpha = 0.0001,\ \alpha = 0.1$. Note that for $\alpha = 0.0001$, the numerical and analytic curves are on top of each other. (B) All curves correspond to analytic values for different choice of parameters $(\alpha,\ \kappa,\ b)$, namely, $(0.0001,\ 0.05,\ 0.01)$, $(0.0001,\ 0.10,\ 0.01)$, $(0.1,\ 0.05,\ 0.01)$, $(0.1,\ 0.10,\ 0.01)$, $(0.1,\ 0.05,\ 0.02)$, $(0.1,\ 0.10,\ 0.02)$, $(0.1,\ 0.15,\ 0.02)$(upper to lower).

3.4.2 The condensation operator and critical exponent

In this section, we shall investigate the effect of BI coupling parameter and backreaction on the condensation operator near critical temperature for Einstein and GB gravity. To investigate the critical exponent, we compute the field eq.(3.86) for $\phi(z)$ near to the critical temperature (T_c) substituting eq.(3.103)

$$\phi''(z) - \frac{d-4}{z}\phi'(z) + \frac{d-2}{r_+^2} b\phi'(z)^3 z^3 = \frac{\langle J\rangle^2}{r_+^2}\mathcal{B}(z)\phi(z) \tag{3.131}$$

where $\mathcal{B}(z) = \frac{2z^{2\Delta_+-4}}{r_+^{2\Delta_+-4}} \frac{F^2(z)}{f(z)} \left(1 - \frac{bz^4}{r_+^2}\phi'(z)^2\right)^{\frac{3}{2}}$.

For general study, we have considered here $f(z)$ as GB gravity metric, because it reduces to Einstein gravity metric when GB parameter (α) becomes zero. We may now expand $\phi(z)$ in the small parameter $\frac{\langle J\rangle^2}{r_+^2}$ as

$$\frac{\phi(z)}{r_+} = \lambda\left\{(1 - z^{d-3}) - \frac{b(\lambda^2|_{b=0})(d-3)^3}{2(3d-7)}(1 - z^{3d-7})\right\} + \frac{\langle J\rangle^2}{r_+^2}\zeta(z) \qquad (3.132)$$

with $\zeta(1) = 0 = \zeta'(1)$.

Using eq.(3.132) and comparing the coefficient of $\frac{\langle J\rangle^2}{r_+^2}$ of left hand side and right hand side of the eq.(3.131) (keeping term upto $\mathcal{O}(b)$), we get the equation for the correction $\zeta(z)$ near to the critical temperature

$$\zeta''(z) - \left\{\frac{d-4}{z} + 3b(\lambda^2|_{b=0})(d-2)(d-3)^2 z^{2d-5}\right\}\zeta'(z) = \lambda\frac{2z^{2\Delta_+-4}}{r_+^{2\Delta_+-4}}\frac{F^2(z)}{f(z)}\mathcal{A}_1(z) \qquad (3.133)$$

where $\mathcal{A}_1(z) = 1 - z^{d-3} - \frac{3b(\lambda^2|_{b=0})(d-3)^2}{2}\left\{(1 - z^{d-3})z^{2d-4} + \frac{d-3}{3(3d-7)}(1 - z^{3d-7})\right\}$.

To solve this equation, we need to multiply this equation by $z^{-(d-4)}e^{\frac{3(d-2)(d-3)^2 b(\lambda^2|_{b=0})}{2d-4}z^{2d-4}}$.

After multiplying this and using eq.(3.122), eq.(3.133) becomes

$$\begin{aligned}\frac{d}{dz}\left(z^{-(d-4)}e^{\frac{3(d-2)(d-3)^2 b(\lambda^2|_{b=0})}{2d-4}z^{2d-4}}\zeta'(z)\right) &= \lambda\frac{2z^{2\Delta_+-2}}{r_+^{2\Delta_+-2}}\frac{z^{-(d-4)}F^2(z)}{(g_0(z) + (d-4)(d-3)\alpha g_1(z))} \\ &\times\ e^{\frac{3(d-2)(d-3)^2 b(\lambda^2|_{b=0})z^{2d-4}}{2d-4}}\mathcal{A}_1(z) \qquad (3.134)\end{aligned}$$

Using boundary condition of $\zeta(z)$, we integrate (3.134) between the limits $z = 0$ and $z = 1$. Finally we get

$$\frac{\zeta'(z)}{z^{d-4}}\,|_{z\to 0} = -\frac{\lambda}{r_+^{2\Delta_+-2}}\mathcal{A}_2 \qquad (3.135)$$

where $\mathcal{A}_2 = \int_0^1 dz \frac{2z^{2\Delta_+-2}z^{-(d-4)}F^2(z)}{(g_0(z)+(d-4)(d-3)\alpha g_1(z))}e^{\frac{3(d-2)(d-3)^2 b(\lambda^2|_{b=0})}{2d-4}z^{2d-4}}\mathcal{A}_1(z)$.

It is noticed that $\zeta'(z)$ and $(d-3)th$ derivative of $\zeta(z)$ are related by

$$\frac{\zeta^{d-3}(z=0)}{(d-4)!} = \frac{\zeta'(z)}{z^{d-4}}|_{z\to 0} \qquad (3.136)$$

The asymptotic behaviour of $\phi(z)$ is given by eq.(3.81). But from eq.(3.132), we get the asymptotic behaviour (near $z = 0$) of $\phi(z)$. Comparing the both equations of $\phi(z)$ about $z = 0$, we obtain

$$\begin{aligned}\mu - \frac{\rho}{r_+^{d-3}}z^{d-3} &= \lambda r_+\left\{(1 - z^{d-3}) - \frac{b(\lambda^2|_{b=0})(d-3)^3}{2(3d-7)}(1 - z^{3d-7})\right\} \\ &+ \frac{\langle J\rangle^2}{r_+}\left\{\zeta(0) + z\zeta'(0) + \ldots\ldots + \frac{\zeta^{d-3}(0)}{(d-3)!}z^{d-3} + \ldots.\right\} \qquad (3.137)\end{aligned}$$

Comparing the coefficient of z^{d-3} both side of eq.(3.137), we obtain

$$-\frac{\rho}{r_+^{d-3}} = -\lambda r_+ + \frac{\langle J\rangle^2}{r_+}.\frac{\zeta^{d-3}(0)}{(d-3)!} \tag{3.138}$$

Using eq.(3.135) and eq.(3.136), we obtain the relation between the charge density (ρ) and the condensation operator ($\langle J\rangle$) i.e.

$$\frac{\rho}{r_+^{d-2}} = \lambda\left[1 + \frac{\langle J\rangle^2}{r_+^{2\Delta_+}}.\frac{\mathcal{A}_2}{(d-3)}\right] \tag{3.139}$$

Using eq.(3.109) and definition of λ and simplifying eq.(3.139), we get

$$\langle J\rangle^2 = \frac{(d-3)(4\pi T_c)^{2\Delta_+}}{\mathcal{A}_2[(d-1) - \frac{(d-3)^2}{(d-2)}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})]^{2\Delta_+}}.\left(\frac{T_c}{T}\right)^{d-2}\left[1-\left(\frac{T}{T_c}\right)^{d-2}\right]. \tag{3.140}$$

The temperature is away to (but close to) critical temperature (i.e. $T \approx T_c$) that's why we can write as

$$\begin{aligned}\left(\frac{T_c}{T}\right)^{d-2}\left[1-\left(\frac{T}{T_c}\right)^{d-2}\right] &= \left(\frac{T_c}{T}\right)^{d-2}\left[1-\left(\frac{T}{T_c}\right)\right]\left[1+\frac{T}{T_c}+\left(\frac{T_c}{T}\right)^2+.....+\left(\frac{T_c}{T}\right)^{d-3}\right]\\ &= (d-2)\left[1-\left(\frac{T}{T_c}\right)\right]\end{aligned} \tag{3.141}$$

Finally, we obtain the relation between the condensation operator and the critical temperature in general dimension i.e.

$$\langle J\rangle = \beta T_c^{\Delta_+}\sqrt{1-\frac{T}{T_c}} \tag{3.142}$$

where $\beta = \sqrt{\frac{(d-3)(d-2)}{\mathcal{A}_2}}\left[\frac{4\pi}{(d-1)-\frac{(d-3)^2}{(d-2)}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})}\right]^{\Delta_+}$.

From this eq.(3.142), we find that the critical exponent is 1/2 which is not affected by the GB gravity, BI coupling parameter and backreactions. This equation is valid for $d = 4, 5$ dimension and is consistence with previous finding result for different value of m^2 for Einstein gravity and GB gravity.
Here we explicitly compute our rest of calculation with $d = 5,\ m^2 = -3$. The choice for m^2 yields $\Delta_+ = 3$. The eq.(3.142) becomes

$$\langle J\rangle = \beta T_c^3\sqrt{1-\frac{T}{T_c}} \tag{3.143}$$

Now $\mathcal{A}_1(z),\ \mathcal{A}_2$ and β become in $d = 5$-dimension respectively,

$$\begin{aligned}\mathcal{A}_1(z) &= 1-z^2-\frac{b(\lambda^2|_{b=0})}{2}(1-z^2)\left[12z^6+\frac{1-z^8}{1-z^2}\right]\\ &= (1-z^2)\left[1-\frac{b(\lambda^2|_{b=0})}{2}(1+z^2+z^4+13z^6)\right]\\ \mathcal{A}_2 &= \int_0^1 dz\frac{2z^3F^2(z)}{(g_0(z)+2\alpha g_1(z))}e^{6b(\lambda^2|_{b=0})z^6}\mathcal{A}_1(z)\\ \beta &= \sqrt{\frac{6}{\mathcal{A}_2}}\left[\frac{\pi}{1-\frac{1}{3}\kappa_i^2(\lambda^2|_{\kappa_{i-1}})}\right]^3.\end{aligned} \tag{3.144}$$

In $\mathcal{A}_2$, we can expand $e^{6b(\lambda^2|_{b=0})z^6} = 1 + 6b(\lambda^2|_{b=0})z^6 + \mathcal{O}(b^2)$ in the interval $[0, 1)$. Simplifying $\mathcal{A}_2$ upto $\mathcal{O}(b)$, we obtain

$$\mathcal{A}_2 = \int_0^1 dz \frac{2z^3F^2(z)(1-z^2)}{(g_0(z)+2\alpha g_1(z))} \left\{1 - \frac{b(\lambda^2|_{b=0})}{2}(1+z^2+z^4+z^6)\right\} \quad (3.145)$$

In Einstein gravity, the metric term should be $g_0(z)$ (because $\alpha = 0$). Using eq.(3.100) and eq.(3.108) and computing $\mathcal{A}_2$ with $\tilde{\alpha} = 0.7218$ for $\kappa = 0,\ b = 0$, we obtain $\beta = 238.908$ which is very good agreement with the exact result $\beta = 238.958$ [144].

Now we shall proceed to included the effect of BI parameter ($b \neq 0$) and backreaction ($\kappa \neq 0$) in our finding . For $\kappa = 0,\ b = 0.01$, computing $\mathcal{A}_2$ with $\tilde{\alpha} = 0.7540$, we get $\beta = 270.834$ which is good agreement with the exact result $\beta = 271.612$ [144]. For $\kappa = 0.05,\ b = 0$, computing $\mathcal{A}_2$ with $\tilde{\alpha} = 0.7195$, we get $\beta = 248.959$. In the Table 3.7, we present the analytic results for Einstein gravity.

Table 3.7: The analytical results for the condensation operator with backreaction and Born-Infeld parameter in Einstein gravity ($\alpha = 0$)

κ	b	$\tilde{\alpha}$	λ^2_{SL}	$\mathcal{A}_2$	$\beta\|_{SL} = \frac{\langle J \rangle}{T_c^3\sqrt{1-T/T_c}}$
0.0	0.0	0.7218	18.23	0.101062	238.908
0.0	0.01	0.7540	25.91	0.07864	270.834
0.05	0.0	0.7195	18.11	0.10202	248.959
0.05	0.01	0.7525	25.68	0.07938	287.815
0.10	0.0	0.7122	17.75	0.10508	282.416
0.10	0.01	0.7455	25.02	0.08203	346.824

In Gauss-Bonnet gravity, we use the metric form from eq.(3.122) and eq.(3.123). Set GB parameter $\alpha = 0.1$ and $\kappa = 0$, computing $\mathcal{A}_2$ with $\tilde{\alpha} = 0.7080$ for $b = 0$, we obtain $\beta = 244.112$ which is very good agreement with the exact result $\beta = 243.897$ [144]. For $b = 0.01,\ \tilde{\alpha} = 0.7665$ with same setting, we obtain $\beta = 294.147$ which is good agreement with exact $\beta = 290.107$ [144]. In the table 3.8, we present the analytic result of condensation operator for GB gravity.

In Fig. 3.4, the plot of $\frac{\langle J \rangle}{T_c^3}$ vs. $\frac{T}{T_c}$ is shown for Einstein gravity and GB gravity for different choice of $\kappa,\ b$.

Table 3.8: The analytical results for the condensation operator with backreaction and Born-Infeld parameter in GB gravity ($\alpha = 0.1$)

κ	b	$\tilde{\alpha}$	λ^2_{SL}	$\mathcal{A}_2$	$\beta\|_{SL} = \frac{\langle J \rangle}{T_c^3\sqrt{1-T/T_c}}$
0.0	0.0	0.7080	24.18	0.096798	244.112
0.0	0.01	0.7665	39.96	0.066669	294.147
0.05	0.0	0.7053	23.96	0.098019	257.863
0.05	0.01	0.7645	39.42	0.067623	323.298
0.10	0.0	0.6935	23.30	0.10255	304.436
0.10	0.01	0.7505	37.88	0.071655	432.956

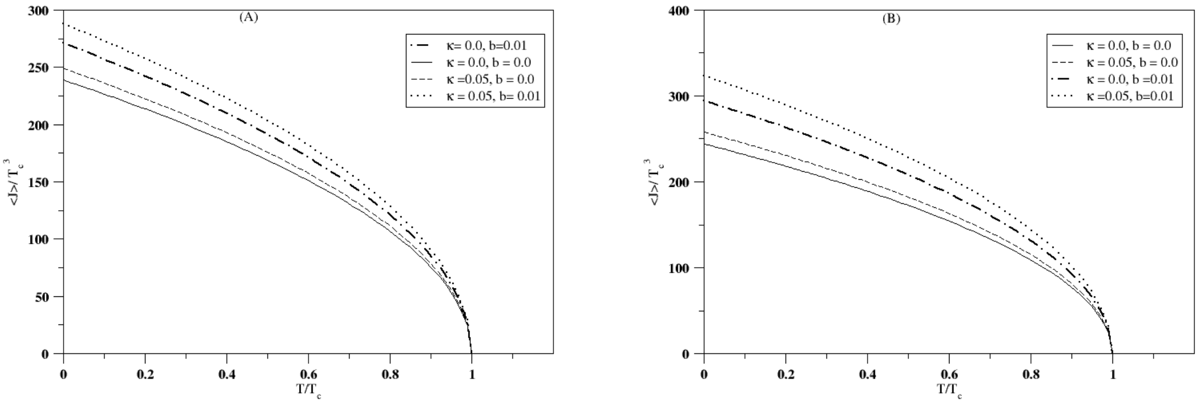

Figure 3.4: (A) $\langle J\rangle/T_c^3$ vs. T/T_c plot for Einstein holographic superconductors for different choice of parameters $\kappa,\ b$. (B) $\langle J\rangle/T_c^3$ vs. T/T_c plot for Gauss Bonnet holographic superconductors with GB parameter $\alpha = 0.1$ for different choice of parameters $\kappa,\ b$.

3.5 Conclusions

In this chapter, we have first discussed briefly superconductivity and the basics of holographic superconductors. We then mention about the motivation of our work on holographic superconductors in the presence of non-linear electrodynamics with background of Einstein gravity and Gauss-Bonnet gravity. In our investigation [35], we have shown the analytical relation between the critical temperature and the charge density of higher dimensional holographic superconductors in the framework of Born-Infeld electrodynamics taking into account the effect of backreaction of the matter fields on the spacetime metric. In particular, the relation between the critical temperature and the charge density holds for a d-dimensional holographic superconductor. We observe that the condensation gets hard to form in the presence of the Born-Infeld parameter. It is also noted that the condensate gets harder to form in Gauss-Bonnet gravity than Einstein gravity in $4+1$-dimensions. The inclusion of the effect of backreaction of the matter fields on the spacetime metric makes the condensate even harder to form. We find that our results are in very good agreement with the existing numerical results [150, 149]. We would now like to mention the importance of our results obtained analytically. It is evident that the Sturm-Liouville eigenvalue method is a powerful analytical approach to investigate holographic superconductors taking into account the effect of various parameters, namely, the Born-Infeld parameter and the Gauss-Bonnet coupling parameter. One of the great advantages of this approach is that it is also found to be applicable away from the probe limit. This can be inferred by comparing the analytical results with the numerical results. It should also be appreciated that the analytical method is always more reliable than the numerical approach since the reliability of the numerical results decreases when the temperature T approaches to zero [75]. We further point out that our analytical results obtained by the Sturm-Liouville eigenvalue method also agree with the results obtained from an alternative analytic technique known as the matching method [149]. Our general result presented in d-dimensions can also be applied for values of $d \geq 4$ for Einstein gravity and $d \geq 5$ for Gauss-Bonnet gravity.

Chapter 4

Holographic free energy and thermodynamic geometry

4.1 Introduction

Black hole thermodynamics was first introduced by Bekenstein [52] and Hawking [53] in the 1970s, in which a relation between thermodynamical variables and geometrical quantities of black hole had been established. Thermodynamic structure of black hole was formulated by establishing four laws of black hole thermodynamics. Phase transition, a basic phenomenon in thermodynamics is observed in the case of black hole thermodynamics [70]. These transitions are related to the transition between one black hole spacetime to another black hole spacetime or thermal state of spacetime. To analyze thermodynamical properties, a number of methods had been developed in ordinary thermodynamics.

The graphical and geometrical method to analyse a system in thermodynamical equilibrium was first introduced by Gibbs [153]. The internal energy, entropy and volume of the system are labeled as coordinate system of the geometry which is known as Gibbs space. The geometry of this thermodynamic phase space does not have an intrinsic metric structure. To introduce an intrinsic metric structure, Weinhold [154, 155] proposed a different form of geometry which is isomorphic to an ordinary Euclidean space. This geometry is associated with an abstract metric space which is constructed from general thermodynamical principles. The thermodynamic laws are represented by the axioms of the abstract metric space which allows certain advantage to analyse a system in thermodynamical equilibrium. This implies that a thermodynamical equilibrium state is associated with the geometry of a metric space. The advantage of this formalism is that one can generalize some thermodynamical relations (like Gibbs-Duhem) in more abstract way by eliminating different assumptions (scaling hypothesis, homogeneous potential). Underlying metric structure of this geometry incorporates all basic details of a system in thermodynamical equilibrium. Weinhold [154, 155] defined metric as the Hessian of the internal energy. Later Ruppeiner [156] defined a similar type of metric which is related to the Weinhold metric with the conformal factor. Although the formulation of Weinhold's approach and Ruppeiner's approach are not Legendre invariant, the

Riemanian scalar curvature for Ruppeiner's approach always diverges at the critical point of phase transition. In other words, the divergence of scalar curvature indicates the phase transition point. The physical interpretation of this scalar curvature is that it captures the information about phase transition and any interaction of the thermodynamical system. The Ruppeiner's scalar curvature R [156] vanishes for non-interacting thermodynamical system whereas the non-zero value of R indicates interacting thermodynamical system. This geometrical method is also very helpful to study black hole phase transition or phase transition in gravity theory. It turns out that this framework based on a geometrical structure gives a handle to study critical phenomena [156].

In this chapter, we set out to study the critical phenomena of holographic superconductors in presence of Maxwell electrodynamics and Born-Infeld electrodynamics using the formalism of the thermodynamic geometry. We employ the matching method [74] to obtain the behaviour of the matter fields near the horizon of the black hole. This in turn is used to compute the critical temperature and the condensation operator. In the first section, we consider Maxwell's electrodynamics and we obtain the critical temperature for a set of values of the matching point where the near horizon and boundary behaviour of the fields are matched. The analysis is based on the probe limit approximation (which neglects the back reaction of the matter fields on the background spacetime geometry) and is carried out for two sets of boundary conditions for the condensation operator, namely $\psi_- \neq 0$, $\psi_+ = 0$ and $\psi_+ \neq 0$, $\psi_- = 0$. We then proceed to compute the free energy of this $2+1$-dimensional holographic superconductor. The trick here is to relate the free energy of the theory on the boundary to the value of the on-shell action of the Abelian-Higgs sector of the full Euclidean action with proper boundary terms [157],[158]. From this, we compute the thermodynamic metric using the formalism of [156]. The computation is once again carried out for both sets of boundary conditions for the condensation operator as mentioned earlier. The scalar curvature is computed next and the temperature at which the scalar curvature diverges is said to be the critical temperature in this approach. This temperature is then compared with that obtained from the matching method.
In the next section, we exploit the formalism of thermodynamic geometry once again to investigate the properties of $2+1$-dimensional holographic superconductors in presence of Born-Infeld electrodynamics. However, this time our intention is to study the effects of non-linearity in such systems using this formalism. The non-linearity is introduced by coupling the charged scalar field to Born-Infeld (BI) electrodynamics. There has been a lot of work in which holographic superconductors has been investigated in the presence of BI electrodynamics [140]-[146]. It would therefore be interesting to assess these results with the results obtained from the analysis in this work. We follow similar procedure to obtain the critical temperature and the condensation values. The study is carried out for the boundary condition $\psi_+ \neq 0$, $\psi_- = 0$ for the condensation operator. We then calculate the free energy of this $2+1$-dimensional holographic superconductor in the presence of BI electrodynamics. It is observed that the free energy of the holographic superconductor on the boundary gets corrections due to the BI parameter thereby capturing the effects of

non-linearity. Therefore, the thermodynamic metric computed from this free energy also capture the effects of non-linearity. This is then exploited to calculate the critical temperature from the divergence of the scalar curvature. We finally compare our findings with those obtained from the matching method. The analysis gives us yet another way of comparing the results with those obtained from other analytical techniques, namely, the Sturm-Liouville eigenvalue method and the matching method.

4.2 Basic set up

The plane-symmetric black hole metric can be assumed to take the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \tag{4.1}$$

where $f(r) = r^2\left(1 - \frac{r_+^3}{r^3}\right)$ and r_+ is the horizon radius.
The Hawking temperature of this black hole reads

$$T_h = \frac{f'(r_+)}{4\pi} = \frac{3}{4\pi}r_+ \ . \tag{4.2}$$

This temperature is interpreted as the temperature of the conformal field theory on the boundary.
In 3+1-dimensions, the action for the model of a holographic superconductor consists a complex scalar field coupled to a $U(1)$ gauge field in anti-de Sitter spacetime

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R - 2\Lambda) + \mathcal{L}_e - (D_\mu\psi)^* D^\mu\psi - m^2\psi^*\psi\right] \tag{4.3}$$

where $\Lambda = -\frac{3}{L^2}$, $\kappa^2 = 8\pi G$ and $\mathcal{L}_e$ denotes Lagrangian density of the electrodynamics. In this chapter, we are working with two type of electrodynamics, namely, Maxwell electrodynamics ($\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$) and Born-Infeld electrodynamics ($\mathcal{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{b}{2}F^{\mu\nu}F_{\mu\nu} - \frac{b^2}{16}(G^{\alpha\beta}F_{\alpha\beta})^2}\right)$). To study critical phenomena of holographic superconductors, we make the ansatz for gauge and matter field as

$$A_\mu = (\phi(r), 0, 0, 0) \ , \quad \psi = \psi(r) \ . \tag{4.4}$$

Regularization of the fields and the aysmptotic behaviour of the fields are same as mentioned in earlier chapter. For $3 + 1$ dimensions, the matter fields obey [74] near boundary of the spacetime

$$\phi_b(r) = \mu - \frac{\rho}{r} \tag{4.5}$$

$$\psi_b(r) = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}} \tag{4.6}$$

where the conformal dimension $\Delta_\pm$ is given by

$$\Delta_\pm = \frac{3 \pm \sqrt{9 + 4m^2}}{2} \ . \tag{4.7}$$

The parameters μ and ρ are interpreted to be dual to the chemical potential and charge density of the conformal field theory on the boundary. We now make change of coordinates $z = \frac{1}{r}$. Under this transformation, the plane-symmetric black hole metric (4.1) and the Hawking temperature (4.2) take the form

$$ds^2 = \frac{1}{z^2}\left(-F(z)dt^2 + \frac{dz^2}{F(z)} + dx^2 + dy^2\right) \quad ; \quad f(z) = \frac{1}{z^2}F(z) \tag{4.8}$$

$$T_h = \frac{3}{4\pi z_h} \tag{4.9}$$

where $F(z) = (1 - \frac{z^3}{z_h^3})$ and $z_h = \frac{1}{r_+}$. In z-coordinate, the regularization condition $\phi(r_+) = 0$ translates to $\phi(z = z_h) = 0$ and the asymptotic behaviour of the fields read

$$\phi_b(z) = \mu - \rho z \tag{4.10}$$

$$\psi_b(z) = \psi_- z^{\Delta_-} + \psi_+ z^{\Delta_+} \ . \tag{4.11}$$

These boundary conditions are need to solve field equations in both cases ($\mathcal{L}_e = \mathcal{L}_M, \mathcal{L}_{BI}$). Let proceed to investigate the critical temperature and the condensation in presence of Maxwell's eletrodynamics.

4.3 In presence of Maxwell electrodynamics

This section is based on the investigation in [36]. In presence of Maxwell electrodynamics, the Lagragian density of electrodynamics becomes $\mathcal{L}_e = \mathcal{L}_M$ and the action (4.3) reads

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R - 2\Lambda) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu\psi)^* D^\mu\psi - m^2\psi^*\psi\right] \ . \tag{4.12}$$

The ansatz (4.4) leads to the following equations of motion for the matter fields

$$\phi''(r) + \frac{2}{r}\phi'(r) - \frac{2q^2\psi^2(r)\phi(r)}{f(r)} = 0 \tag{4.13}$$

$$\psi''(r) + \left(\frac{2}{r} + \frac{f'(r)}{f(r)}\right)\psi'(r) + \left(\frac{q^2\phi^2(r)}{f(r)^2} - \frac{m^2}{f(r)}\right)\psi(r) = 0 \tag{4.14}$$

where prime denotes derivative with respect to r. We set $q = 1$ for rest our calculation as mentioned in earlier chapter. In z-coordinate, the field eq.(s) (4.13),(4.14) become

$$\phi''(z) - \frac{2\psi^2(z)}{z^2F(z)}\phi(z) = 0 \tag{4.15}$$

$$\psi''(z) + \left(\frac{F'(z)}{F(z)} - \frac{2}{z}\right)\psi'(z) + \left(\frac{\phi^2(z)}{F^2(z)} - \frac{m^2}{z^2F(z)}\right)\psi(z) = 0 \tag{4.16}$$

where prime now denotes derivative with respect to z. We now employ the matching method in the interval $(0, z_h)$ to obtain the critical temperature below which the scalar field condensation takes place.

4.3.1 Critical temperature from matching method

To apply the matching method, we require the fields to be finite at the horizon. The Taylor series expansions of these fields near the horizon read

$$\phi_h(z) = \phi(z_h) + \phi'(z_h)(z - z_h) + \frac{\phi''(z_h)}{2}(z - z_h)^2 + ... \tag{4.17}$$

$$\psi_h(z) = \psi(z_h) + \psi'(z_h)(z - z_h) + \frac{\psi''(z_h)}{2}(z - z_h)^2 + ... \tag{4.18}$$

To compute the undetermined coefficients, we use the boundary condition $\phi(z_h) = 0$ along with $f(z_h) = 0$ and eq.(s)(4.15),(4.16). This yields

$$\phi''(z_h) = -\frac{2}{3z_h}\phi'(z_h)\psi^2(z_h) \tag{4.19}$$

$$\psi'(z_h) = -\frac{m^2}{3z_h}\psi(z_h) \quad ; \quad \psi''(z_h) = \frac{\psi(z_h)}{18z_h^2}\left[m^4 + 6m^2 - z^4\phi'^2(z_h)\right] \tag{4.20}$$

In the rest of our analysis, we shall set $m^2 = -2$ which in turn implies $\Delta_- = 1$ and $\Delta_+ = 2$. This is consistent with the Breitenlohner-Freedman bound[94],[95]. Hence the near horizon expansions of these fields upto $\mathcal{O}(z^2)$ read

$$\phi_h(z) = \phi'(z_h)\left[(z - z_h) - \frac{\psi^2(z_h)}{3z_h}(z - z_h)^2\right] \tag{4.21}$$

$$\psi_h(z) = \psi(z_h)\left[1 + \frac{2}{3z_h}(z - z_h) - \frac{(8 + z_h^4\phi'^2(z_h))}{36z_h^2}(z - z_h)^2\right] . \tag{4.22}$$

The matching method involves matching the near horizon expression of the fields with the asymptotic solution of these field at any arbitrary point between the horizon and the boundary, say $z = \frac{z_h}{2}$. In our analysis, we shall match the solution at $z = \frac{z_h}{\lambda}$, where λ lies between $[1, \infty]$. We shall later on set specific values of λ. The matching conditions are

$$\phi_h\left(\frac{z_h}{\lambda}\right) = \phi_b\left(\frac{z_h}{\lambda}\right) \quad ; \quad \phi'_h\left(\frac{z_h}{\lambda}\right) = \phi'_b\left(\frac{z_h}{\lambda}\right) \tag{4.23}$$

$$\psi_h\left(\frac{z_h}{\lambda}\right) = \psi_b\left(\frac{z_h}{\lambda}\right) \quad ; \quad \psi'_h\left(\frac{z_h}{\lambda}\right) = \psi'_b\left(\frac{z_h}{\lambda}\right) . \tag{4.24}$$

From eq.(4.23), we obtain the following relations

$$\psi^2(z_h) = \frac{3\lambda}{1 - \lambda^2}\left(\frac{\mu}{z_h\phi'(z_h)} + 1\right) \tag{4.25}$$

$$\rho = \frac{\lambda}{z_h(\lambda + 1)}\left[2\mu + z_h\phi'(z_h)\left(1 - \frac{1}{\lambda}\right)\right] . \tag{4.26}$$

Similarly, from eq.(4.24), we get

$$\psi_{-/+} = \left[1 - \frac{(\lambda - 1)}{3\lambda}\frac{(3\lambda\Delta + 2)}{\Delta(\lambda - 1) + 2}\right]\left(\frac{\lambda}{z_h}\right)^{\Delta}\psi(z_h) \tag{4.27}$$

$$\phi'^2(z_h) = \frac{1}{z_h^4}\left[\frac{12\lambda}{(\lambda - 1)}\frac{\lambda\Delta - 2(1 - \Delta)}{\Delta(\lambda - 1) + 2} - 8\right] \quad \Rightarrow \phi'(z_h) = -\frac{\chi(\lambda, \Delta)}{z_h^2} \tag{4.28}$$

where

$$\chi(\lambda, \Delta) = \sqrt{\frac{12\lambda}{(\lambda-1)}\frac{\lambda\Delta - 2(1-\Delta)}{\Delta(\lambda-1)+2} - 8}\ . \tag{4.29}$$

In eq.(4.27), ψ_- is for $\Delta = \Delta_- = 1$ and ψ_+ is for $\Delta = \Delta_+ = 2$. Note that we consider the negative sign before the square root of $\phi'(z_h)$ because $\phi'(z_h)$ is the electric field due to the charge of the black hole.
Substituting $\phi'(z_h)$ from eq.(4.28) in eq.(s)(4.25),(4.26), we obtain

$$\psi(z_h) = \sqrt{\frac{3\lambda}{\lambda^2-1}\left(\frac{\mu z_h}{\chi(\lambda,\Delta)} - 1\right)} \tag{4.30}$$

$$\rho = \frac{\lambda}{z_h(\lambda+1)}\left[2\mu - \frac{\chi(\lambda,\Delta)}{z_h}\left(1 - \frac{1}{\lambda}\right)\right]. \tag{4.31}$$

Using eq.(s) (4.25)-(4.28) and $T = \frac{3}{4\pi z_h}$, we obtain the condensation operator and the critical temperature in terms of the chemical potential and the charge density

$$\langle\mathcal{O}\rangle = \gamma_{(\mu)} T_c^{\Delta}\left(1 - \frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = \xi_{(\mu)}\mu \tag{4.32}$$

$$\langle\mathcal{O}\rangle = \gamma_{(\rho)} T_c^{\Delta}\left(1 - \frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = \xi_{(\rho)}\sqrt{\rho} \tag{4.33}$$

where

$$\gamma_{(\mu)} = \sqrt{\frac{3\lambda^2}{\lambda^2-1}}\left[1 - \frac{(\lambda-1)}{3\lambda}\frac{(3\lambda\Delta+2)}{\Delta(\lambda-1)+2}\right]\left(\frac{4\pi\lambda}{3}\right)^{\Delta}; \xi_{(\mu)} = \frac{3}{4\pi\chi(\lambda,\Delta)} \tag{4.34}$$

$$\gamma_{(\rho)} = \sqrt{\frac{3\lambda}{\lambda-1}}\left[1 - \frac{(\lambda-1)}{3\lambda}\frac{(3\lambda\Delta+2)}{\Delta(\lambda-1)+2}\right]\left(\frac{4\pi\lambda}{3}\right)^{\Delta}; \xi_{(\rho)} = \frac{3}{4\pi\sqrt{\chi(\lambda,\Delta)}}\ . \tag{4.35}$$

Eq.(s) (4.32)-(4.35) presents the general expressions relating the condensation operator and the critical temperature in terms of the parameter λ. We shall now study the above equations for two different boundary conditions, namely, $\psi_- \neq 0,\ \psi_+ = 0$ and $\psi_+ \neq 0,\ \psi_- = 0$. Let us first consider the case $\psi_- = \langle\mathcal{O}\rangle,\ \psi_+ = 0$. As mentioned earlier setting $\psi_+ = 0$ implies that the condensate ψ_- forms in the absence of the source term ψ_+. For simplicity, we choose the matching point to be the middle point between the horizon and the boundary, that is $z = \frac{z_h}{2}$ which implies setting $\lambda = 2$ in the above expressions. This gives $\chi = \sqrt{8}$ for the value of $\Delta = \Delta_- = 1$. Hence the relation between the condensation operator and the critical temperature reads

$$\langle\mathcal{O}_-\rangle_{(\mu)} = \frac{80\pi}{27} T_c\left(1 - \frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.084\mu \tag{4.36}$$

$$\langle\mathcal{O}_-\rangle_{(\rho)} = \frac{40\pi\sqrt{6}}{27} T_c\left(1 - \frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.142\sqrt{\rho}\ . \tag{4.37}$$

We also carry out our calculations for other values of the matching point by choosing different values of λ. For example, $z = 0.10z_h$ (that is $\lambda = 10$), we obtain $T_c = 0.169\sqrt{\rho}$ which agrees fairly well with the numerical result $T_c = 0.225\sqrt{\rho}$ [75]. For the other case, that is $\psi_+ = \langle\mathcal{O}\rangle$, $\psi_- = 0$, we once again choose the matching point to be the middle point between the horizon and the boundary, that is $z = \frac{z_h}{2}$. This gives $\chi = \sqrt{28}$. Hence the relation between the condensation operator and the critical temperature reads

$$\langle\mathcal{O}_+\rangle_{(\mu)} = \frac{160\pi^2}{27}T_c^2\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.0451\mu \tag{4.38}$$

$$\langle\mathcal{O}_+\rangle_{(\rho)} = \frac{80\pi^2\sqrt{6}}{27}T_c^2\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.104\sqrt{\rho}\,. \tag{4.39}$$

As before we carry out our analysis for other values of the matching point by choosing different values of λ. For example, $z = 0.33z_h$ (that is $\lambda = 3$), we obtain $T_c = 0.119\sqrt{\rho}$ which is in very good agreement with the numerical value of $T_c = 0.118\sqrt{\rho}$ [34]. The analytical results for different values of λ are presented in Tables 4.1 and 4.2. We therefore observe that the results from the matching method depends crucially on the matching point. We would like to mention that there is apriori no way to determine a suitable mathing point. This is a lacuna of this method in contrast to the more analytically sound Sturm-Liouville approach [75], [129].

4.3.2 Free energy of the holographic superconductor

We now proceed to compute the free energy at a finite temperature of the field theory living on the boundary of the $3+1$- bulk theory. To do this the holographic approach is used in relating the free energy (Ω) of the boundary field theory to the product of the temperature (T) and the on-shell value of the Abelian-Higgs [157] sector of the Euclidean action (S_E).
To proceed further, we first write down the action for the Abelian-Higgs sector

$$S_M = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu\psi)^*D^\mu\psi - m^2\psi^*\psi\right]\,. \tag{4.40}$$

Using the ansatz, $m^2 = -2$ and $q = 1$, we get

$$S_M = \int d^4x\left[\frac{\phi'^2(z)}{2} - \frac{F(z)\psi'^2(z)}{z^2} + \frac{\phi^2(z)\psi^2(z)}{z^2F(z)} + \frac{2\psi^2(z)}{z^4}\right]\,. \tag{4.41}$$

Applying the boundary condition ($\phi(z_h) = 0$) and the equations of motion (4.15),(4.16), we obtain the on-shell value of the action S_E to be

$$S_o = \int d^3x\left[-\frac{1}{2}\phi(z)\phi'(z)\,|_{z=0} \;+\; \frac{F(z)\psi(z)\psi'(z)}{z^2}\,|_{z=0} \;-\; \int_0^{z_h} dz\frac{\phi^2(z)\psi^2(z)}{z^2F(z)}\right] \tag{4.42}$$

Substituting the asymptotic behaviour of $\phi(z)$ and $\psi(z)$ in the above action, we get

$$S_o = \int d^3x\left[\frac{\mu\rho}{2} + 3\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)|_{z=0} \;-\; \int_0^{z_h} dz\frac{\phi^2(z)\psi^2(z)}{z^2(1-z^3/z_h^3)}\right]\,. \tag{4.43}$$

Note that the term $\left(\frac{\psi_-^2}{z}\right)|_{z=0}$ diverges and therefore one needs to add a counter term at the boundary to cancel this divergence. For the boundary condition $\psi_- = 0$, $\psi_+ \neq 0$, the counter term comes from the counter action [157], [158]

$$S_c = -\int d^3x \left(\sqrt{-h}\psi^2(z)\right)|_{z=0} \tag{4.44}$$

where h is the determinant of the induced metric on the AdS boundary. Using the asymptotic behaviour of $\psi(z)$ (eq.(4.11)) and evaluating this term, we obtain

$$S_c = -\int d^3x \left[2\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)|_{z=0}\right] . \tag{4.45}$$

The free energy of the $2+1$-boundary field theory can now be obtained by adding S_o and S_c. This yields

$$\begin{aligned} \Omega &= -T(S_o + S_c) \\ &= \beta T V_2 \left[-\frac{\mu\rho}{2} - \psi_+\psi_- + I\right] \\ &= \beta T V_2 \left[-\frac{\mu\rho}{2} + I\right] \end{aligned} \tag{4.46}$$

where in the second equality we have set $\int d^3x = \beta V_2$, V_2 being the volume of the 2-dimensional space of the boundary and in the last equality we have used the fact that $\psi_- = 0$. The integral I reads

$$I = \int_0^{z_h} dz \frac{\phi^2(z)\psi^2(z)}{z^2(1-z^3/z_h^3)} = \int_0^{z_h/\lambda} dz \frac{\phi_b^2(z)\psi_b^2(z)}{z^2(1-z^3/z_h^3)} + \int_{z_h/\lambda}^{z_h} dz \frac{\phi_h^2(z)\psi_h^2(z)}{z^2(1-z^3/z_h^3)} \equiv \mathcal{I}_1 + \mathcal{I}_2 . \tag{4.47}$$

For the other boundary condition, namely, $\psi_+ = 0$, $\psi_- \neq 0$, the counter action reads [157], [158]

$$\begin{aligned} S_c &= -\int d^3x \left(\sqrt{-h} z\psi(z)\psi'(z)\right)|_{z=0} \\ &= -\int d^3x \left[3\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)|_{z=0}\right] . \end{aligned} \tag{4.48}$$

This cancels the divergence term of the on-shell action (4.43) and yields

$$\Omega = \beta T V_2 \left[-\frac{\mu\rho}{2} + I\right] . \tag{4.49}$$

To evaluate the integral, we rewrite the matter field in the following form

$$\psi(z_h) = \chi_1 \sqrt{\frac{\mu z_h}{\chi} - 1} \quad ; \quad \psi_{-/+} = \chi_2 \frac{\psi(z_h)}{z_h^\Delta} \tag{4.50}$$

where $\chi_1 = \sqrt{\frac{3\lambda^2}{\lambda^2-1}}$ and $\chi_2 = \lambda^{\Delta}\left[1 - \frac{(\lambda-1)}{3\lambda}\frac{3\lambda\Delta+2}{\Delta(\lambda-1)+2}\right]$. Now using the substitution $z = z_h l$, we obtain

$$\begin{aligned}
\mathcal{I}_1 = \int_0^{z_h/\lambda} dz \frac{\phi_b^2(z)\psi_b^2(z)}{z^2(1-z^3/z_h^3)} &= \int_0^{z_h/\lambda} dz \frac{(\mu-\rho z)^2\psi_{-/+}^2 z^{2\Delta}}{z^2(1-z^3/z_h^3)} \\
&= \psi_{-/+}^2 z_h^{2\Delta-1}\left[\mu^2\mathcal{A}_1 + \rho^2 z_h^2\mathcal{A}_2 - 2\mu\rho z_h\mathcal{A}_3\right] \\
&= \chi_2^2\frac{\psi^2(z_h)}{z_h}\left[B_1\mu^2 + B_2\frac{\mu}{z_h} + B_3\frac{1}{z_h^2}\right] \\
&= \chi_1^2\chi_2^2\left[C_1\mu^3 + C_2\frac{\mu^2}{z_h} + C_3\frac{\mu}{z_h^2} + C_4\frac{1}{z_h^3}\right] \qquad (4.51)
\end{aligned}$$

where the constants are given by the following relations

$$\begin{aligned}
\mathcal{A}_1 &= \int_0^{1/\lambda}\frac{l^{2\Delta-2}dl}{(1-l^3)} \quad ; \quad \mathcal{A}_2 = \int_0^{1/\lambda}\frac{l^{2\Delta}dl}{(1-l^3)} \quad ; \quad \mathcal{A}_3 = \int_0^{1/\lambda}\frac{l^{2\Delta-1}dl}{(1-l^3)} \\
B_1 &= \mathcal{A}_1 + \frac{4\mathcal{A}_2\lambda^2}{(1+\lambda)^2} - \frac{4\mathcal{A}_3\lambda}{(1+\lambda)} \\
B_2 &= \frac{2\chi\mathcal{A}_3(\lambda-1)}{(1+\lambda)} - \frac{4\chi\mathcal{A}_2(1-1/\lambda)}{(1+1/\lambda)^2} \\
B_3 &= \frac{\mathcal{A}_2\chi^2(\lambda-1)^2}{(\lambda+1)^2} \\
C_1 &= \frac{B_1}{\chi} \quad ; \quad C_2 = \frac{B_2}{\chi} - B_1 \quad ; \quad C_3 = \frac{B_3}{\chi} - B_2 \quad ; \quad C_4 = -B_3 \ . \qquad (4.52)
\end{aligned}$$

The evaluation of the integral $\mathcal{I}_2$ can be done in a similar way and yields

$$\begin{aligned}
\mathcal{I}_2 = \int_{z_h/\lambda}^{z_h} dz\frac{\phi_h^2(z)\psi_h^2(z)}{z^2(1-z^3/z_h^3)} &= \phi'^2(z_h)\psi^2(z_h)z_h\left[\mathcal{A}_4 + \left(\frac{4}{3} - \frac{2\psi^2(z_h)}{3}\right)\mathcal{A}_5\right. \\
&+ \left(\frac{\psi^4(z_h)}{9} + \frac{4}{9} - \frac{8+\chi^2}{18} - \frac{8\psi^2(z_h)}{9}\right)\mathcal{A}_6 + \left(\frac{4\psi^4(z_h)}{27} - \frac{2\psi^2(z_h)}{3}\left(\frac{4}{9} - \frac{8+\chi^2}{18}\right) - \frac{8+\chi^2}{27}\right)\mathcal{A}_7 \\
&+ \left(\frac{(8+\chi^2)^2}{36^2} + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\frac{\psi^4(z_h)}{9} + \frac{2(8+\chi^2)\psi^2(z_h)}{81}\right)\mathcal{A}_8 + \frac{(8+\chi^2)^2\psi^4(z_h)}{9(36)^2}\mathcal{A}_{10} \\
&\left. + \left(\frac{-(8+\chi^2)\psi^4(z_h)}{243} - \frac{2(8+\chi^2)^2\psi^2(z_h)}{3(36)^2}\right)\mathcal{A}_9\right] \qquad (4.53)
\end{aligned}$$

where the constants $\mathcal{A}_n$ $(n = 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10)$ are given by the following relations

$$\mathcal{A}_n = \int_{1/\lambda}^{1}\frac{(l-1)^{n-2}dl}{l^2(1-l^3)} \ . \qquad (4.54)$$

After simplification of the above expression, we get

$$\begin{aligned}
\mathcal{I}_2 &= \phi'^2(z_h)\psi^2(z_h)z_h\left[B_4 + B_5\psi^2(z_h) + B_6\psi^4(z_h)\right] \\
&= \phi'^2(z_h)\psi^2(z_h)z_h\left[C_5 + C_6\mu z_h + C_7\mu^2 z_h^2\right] \\
&= \chi_1^2\chi_3^2\left[D_1\mu^3 + D_2\frac{\mu^2}{z_h} + D_3\frac{\mu}{z_h^2} + D_4\frac{1}{z_h^3}\right] \qquad (4.55)
\end{aligned}$$

where the constants in the above expression are given by the following relations

$$
\begin{aligned}
B_4 &= \mathcal{A}_4 + \frac{4}{3}\mathcal{A}_5 + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\mathcal{A}_6 - \frac{8+\chi^2}{27}\mathcal{A}_7 + \frac{(8+\chi^2)^2}{36^2}\mathcal{A}_8 \\
B_5 &= -\frac{2}{3}\mathcal{A}_5 - \frac{8}{9}\mathcal{A}_6 - \frac{2}{3}\left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\mathcal{A}_7 + \frac{16+2\chi^2}{81}\mathcal{A}_8 - \frac{2}{3}\frac{(8+\chi^2)^2}{36^2}\mathcal{A}_9 \\
B_6 &= \frac{1}{9}\mathcal{A}_6 + \frac{4}{27}\mathcal{A}_7 + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\frac{\mathcal{A}_8}{9} - \frac{8+\chi^2}{243}\mathcal{A}_9 + \frac{(8+\chi^2)^2}{36^2}\frac{\mathcal{A}_{10}}{9} \\
C_5 &= B_4 - \chi_1^2 B_5 + \chi_1^4 B_6 \;\; ; \;\; C_6 = \frac{\chi_1^2}{\chi}B_5 - \frac{2\chi_1^4}{\chi}B_6 \;\; ; \;\; C_7 = \frac{\chi_1^4}{\chi^2}B_6 \\
D_1 &= \frac{C_7}{\chi} \;\; ; \;\; D_2 = \frac{C_6}{\chi} - C_7 \;\; ; \;\; D_3 = \frac{C_5}{\chi} - C_6 \;\; ; \;\; D_4 = -C_5 \;\; .
\end{aligned}
\tag{4.56}
$$

Adding $\mathcal{I}_1$ and $\mathcal{I}_2$, we finally get

$$
I = E_1\mu^3 + E_2\frac{\mu^2}{z_h} + E_3\frac{\mu}{z_h^2} + E_4\frac{1}{z_h^3}
\tag{4.57}
$$

where

$$
\begin{aligned}
E_1 &= \chi_1^2\chi_2^2 C_1 + \chi_1^2\chi_3^2 D_1 \;\; ; \;\; E_2 = \chi_1^2\chi_2^2 C_2 + \chi_1^2\chi_3^2 D_2 \\
E_3 &= \chi_1^2\chi_2^2 C_3 + \chi_1^2\chi_3^2 D_3 \;\; ; \;\; E_4 = \chi_1^2\chi_2^2 C_4 + \chi_1^2\chi_3^2 D_4 \;.
\end{aligned}
\tag{4.58}
$$

Hence the analytical expression for the free energy in terms of the chemical potential reads

$$
\begin{aligned}
\frac{\Omega}{V_2} &= -\frac{\lambda}{\lambda+1}\frac{\mu^2}{z_h} + \frac{(\lambda-1)\chi}{2(\lambda+1)}\frac{\mu}{z_h^2} + I \\
&\equiv G_1\mu^3 + G_2\mu^2 T + G_3\mu T^2 + G_4 T^3
\end{aligned}
\tag{4.59}
$$

where

$$
\begin{aligned}
G_1 &= E_1 \;\; ; \;\; G_2 = \left(E_2 - \frac{\lambda}{\lambda+1}\right)\frac{4\pi}{3} \\
G_3 &= \left(E_3 - \frac{(\lambda-1)\chi}{2(\lambda+1)}\right)\frac{16\pi^2}{9} \;\; ; \;\; G_4 = \frac{64\pi^3}{27}E_4 \;.
\end{aligned}
\tag{4.60}
$$

This expression for the free energy can also be written in terms of the charge density as

$$
\frac{\Omega}{V_2} = H_1\frac{\rho^3}{T^3} + H_2\frac{\rho^2}{T} + H_3\rho T + H_4 T^3
\tag{4.61}
$$

where

$$
\begin{aligned}
H_1 &= \frac{G_1}{8}\left(1+\frac{1}{\lambda}\right)^3\left(\frac{3}{4\pi}\right)^3 \\
H_2 &= \frac{G_2}{4}\left(1+\frac{1}{\lambda}\right)^2\left(\frac{3}{4\pi}\right)^2 + \frac{3\chi G_1}{8}\left(1+\frac{1}{\lambda}\right)^2\left(1-\frac{1}{\lambda}\right)\left(\frac{3}{4\pi}\right) \\
H_3 &= \frac{G_3}{2}\left(1+\frac{1}{\lambda}\right)\left(\frac{3}{4\pi}\right) + \frac{\chi G_2}{2}\left(1-\frac{1}{\lambda^2}\right) + \frac{3\chi^2 G_1}{8}\left(1+\frac{1}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)^2\left(\frac{4\pi}{3}\right) \\
H_4 &= \frac{\chi G_3}{2}\left(1-\frac{1}{\lambda}\right)\left(\frac{4\pi}{3}\right) + \frac{\chi^2 G_2}{4}\left(1-\frac{1}{\lambda}\right)^2\left(\frac{4\pi}{3}\right)^2 + \frac{\chi^3 G_1}{8}\left(1-\frac{1}{\lambda}\right)^3\left(\frac{4\pi}{3}\right)^3 + G_4.
\end{aligned}
\tag{4.62}
$$

In the next section, we shall make use of these results to investigate the thermodynamic geometry of this model in the grand canonical ensemble.

4.3.3 Thermodynamic geometry

With the above results in hand, we now proceed to investigate the thermodynamic geometry of this holographic superconductor. The thermodynamic metric is defined as [154, 155]

$$g_{ij} = -\frac{1}{T}\frac{\partial^2\omega(T,\rho)}{\partial x^i \partial x^j} = -\frac{1}{T}\frac{\partial^2\omega(T,\mu)}{\partial x^i \partial x^j} \tag{4.63}$$

where $\omega = \frac{\Omega}{V_2}$, $x^1 = T$ and $x^2 = \rho$ or μ. Hence the components of the metric in terms of μ read

$$\begin{aligned} g_{TT} &= -\left[2G_3\frac{\mu}{T} + 6G_4\right] \\ g_{T\mu} &= g_{\mu T} = -\left[2G_2\frac{\mu}{T} + 2G_3\right] \\ g_{\mu\mu} &= -\left[6G_1\frac{\mu}{T} + 2G_2\right] \end{aligned} \tag{4.64}$$

and in terms of ρ read

$$\begin{aligned} g_{TT} &= -\left[12H_1\frac{\rho^3}{T^6} + 2H_2\frac{\rho^2}{T^4} + 6H_4\right] \\ g_{T\rho} &= g_{\rho T} = \left[9H_1\frac{\rho^2}{T^5} + 2H_2\frac{\rho}{T^3} - H_3\frac{1}{T}\right] \\ g_{\rho\rho} &= -\left[6H_1\frac{\rho}{T^4} + 2H_2\frac{1}{T^2}\right] . \end{aligned} \tag{4.65}$$

The scalar curvature of a general metric

$$ds^2_{th} = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 \tag{4.66}$$

is given by [156]

$$R = \frac{-1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^1}\right)\right] \tag{4.67}$$

To look for any singularity in R, one has to see whether the denominator of the right hand side of eq.(4.67) vanishes. The condition of the divergence of R is $det g_{ij} = 0$. For the metric (4.64), this gives

$$4\left[(3G_2G_4 - G_3^2) + (9G_1G_4 - G_2G_3)\frac{\mu}{T} + (3G_1G_3 - G_2^2)\frac{\mu^2}{T^2}\right] = 0 \tag{4.68}$$

and for the metric (4.65), this gives

$$\left[-9H_1^2\frac{\rho^4}{T^{10}} + 18H_1H_3\frac{\rho^2}{T^6} + (36H_1H_4 + 4H_2H_3)\frac{\rho}{T^4} + (12H_2H_4 - H_3^2)\frac{1}{T^2}\right] = 0 . \tag{4.69}$$

The temperature for which the scalar curvature vanishes can be obtained by solving these equations. We obtain this critical temperature for the two different boundary conditions $\psi_- \neq 0,\ \psi_+ = 0$ and $\psi_+ \neq 0,\ \psi_- = 0$ for a set of values of λ and compare them with the results which have been obtained from the matching method. The Tables 4.1 and 4.2 give the results for $\Delta = \Delta_+$ and $\Delta = \Delta_-$ obtained from the matching method and the thermodynamic geometry for a set of values of the matching point for these two different boundary conditions respectively.
Considering the boundary condition $\psi_+ \neq 0,\ \psi_- = 0$ (which implies $\Delta = \Delta_+ = 2$) and setting $\lambda = 2$, the above equations yield $T_c = 0.084\mu$. This does not agree with the result in [158]. For $\lambda = 3$ (that is $z = 0.33 z_h$), we obtain $T_c = 0.118\sqrt{\rho}$ from the divergence of the scalar curvature. This turns out to agree very well with the numerical value $T_c = 0.118\sqrt{\rho}$ [34].

Table 4.1: For $\Delta = \Delta_+ = 2$, the critical temperature $T_c = \xi_{(\rho)}\sqrt{\rho}$ with numerical value $\xi_{(\rho)} = 0.118$.

Value of λ	Matching point	From matching method		From divergence of R	
		$\xi_{(\mu)}$	$\xi_{(\rho)}$	$\xi_{(\mu)}$	$\xi_{(\rho)}$
5	0.20 z_h	0.076	0.134	0.126	0.152
3	0.33 z_h	0.060	0.119	0.099	0.118
2	0.50 z_h	0.045	0.104	0.084	0.104
$\frac{3}{2}$	0.66 z_h	0.033	0.089	0.080	0.101
$\frac{5}{4}$	0.80 z_h	0.024	0.076	0.084	0.102

Table 4.2: For $\Delta = \Delta_- = 1$, the critical temperature $T_c = \xi_{(\rho)}\sqrt{\rho}$ with numerical value $\xi_{(\rho)} = 0.225$.

Value of λ	Matching point	From matching method		From divergence of R	
		$\xi_{(\mu)}$	$\xi_{(\rho)}$	$\xi_{(\mu)}$	$\xi_{(\rho)}$
5	0.20 z_h	0.113	0.164	0.334	0.709
3	0.33 z_h	0.102	0.156	0.475	0.340
2	0.50 z_h	0.084	0.142	0.359	0.345
$\frac{3}{2}$	0.66 z_h	0.065	0.124	0.355	0.284
$\frac{5}{4}$	0.80 z_h	0.047	0.106	0.375	0.233

4.4 In presence of Born-Infeld electrodynamics

In this section[1], we shall consider Born-Infeld electrodynamics ($\mathcal{L}_e = \mathcal{L}_{BI}$) instead of Maxwell electrodynamics for which the actions reads

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R - 2\Lambda\right) + \mathcal{L}_{BI} - (D_\mu\psi)^* D^\mu\psi - m^2\psi^*\psi\right] \tag{4.70}$$

[1]This part of the discussion is based on our work [37].

where $\mathcal{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{b}{2}F^{\mu\nu}F_{\mu\nu} - \frac{b^2}{16}(G^{\alpha\beta}F_{\alpha\beta})^2}\right)$.
Since we are working with the ansatz (4.4), the last term in square-root does not contribute in the analysis. This ansatz leads to the following equations of motion for the matter fields [129],

$$\phi''(r) + \frac{2}{r}\phi'(r) - \frac{2}{r}b\phi'^3(r) - \frac{2q^2\psi^2(r)\phi(r)}{f(r)}\left\{1 - b\phi'^2(r)\right\}^{3/2} = 0 \quad (4.71)$$

$$\psi''(r) + \left(\frac{2}{r} + \frac{f'(r)}{f(r)}\right)\psi'(r) + \left(\frac{q^2\phi^2(r)}{f(r)^2} - \frac{m^2}{f(r)}\right)\psi(r) = 0 \quad (4.72)$$

where prime denotes derivative with respect to r. We shall also set $q = 1$ since we shall carry out our analysis in the probe limit [134]. Note that the rescaling of the bulk fields ϕ, ψ and the Born-Infeld coupling parameter b as $\phi \to \frac{\phi}{q}$, $\psi \to \frac{\psi}{q}$, $b \to q^2 b$ puts a factor of $\frac{1}{q^2}$ in front of the matter part of the action (4.70). The probe limit corresponds to $\frac{\kappa^2}{q^2} \to 0$ [75]. Changing the coordinate $z = \frac{1}{r}$, the field eq.(s) (4.71),(4.72) yield

$$\begin{aligned} \phi''(z) + 2bz^3\phi'^3(z) - \frac{2\psi^2(z)\phi(z)}{z^2F(z)}\left\{1 - bz^4\phi'^2(z)\right\}^{3/2} &= 0 \quad (4.73) \\ \psi''(z) + \left(\frac{F'(z)}{F(z)} - \frac{2}{z}\right)\psi'(z) + \left(\frac{\phi^2(z)}{F^2(z)} - \frac{m^2}{z^2F(z)}\right)\psi(z) &= 0\,. \quad (4.74) \end{aligned}$$

To obtain the critical temperature below which the scalar field condensation takes place we employ the matching method in the interval $(0, z_h)$.

4.4.1 Matching method for BI electrodynamics

The Taylor series expansions of the fields near the horizon read

$$\begin{aligned} \phi_h(z) &= \phi(z_h) + \phi'(z_h)(z - z_h) + \frac{\phi''(z_h)}{2}(z - z_h)^2 + ... \quad (4.75) \\ \psi_h(z) &= \psi(z_h) + \psi'(z_h)(z - z_h) + \frac{\psi''(z_h)}{2}(z - z_h)^2 + ... \quad (4.76) \end{aligned}$$

We shall now determine the undetermined coefficients using the boundary condition $\phi(z_h) = 0$ along with $f(z_h) = 0$ and eq.(s)(4.73),(4.74). This yields upto first order in the Born-Infeld parameter

$$\begin{aligned} \phi''(z_h) &= -\left[2bz_h^3\phi'^3(z_h) + \frac{2\phi'(z_h)\psi^2(z_h)}{3z_h}\left(1 - \frac{3}{2}bz_h^4\phi'^2(z_h)\right)\right] \quad (4.77) \\ \psi'(z_h) &= -\frac{m^2}{3z_h}\psi(z_h) \quad ; \quad \psi''(z_h) = \frac{\psi(z_h)}{18z_h^2}\left[m^4 + 6m^2 - z^4\phi'^2(z_h)\right]. \quad (4.78) \end{aligned}$$

This is consistent with the Breitenlohner-Freedman bound [94],[95]. Hence the near horizon expansions of these fields upto $\mathcal{O}((z-z_h)^2)$ read

$$\phi_h(z) = \phi'(z_h)\left[(z-z_h) - \frac{\psi^2(z_h)}{3z_h}(z-z_h)^2\right] + b\phi'^3(z_h)z_h^3(z-z_h)^2\left[\frac{\psi^2(z_h)}{2} - 1\right] \tag{4.79}$$

$$\psi_h(z) = \psi(z_h)\left[1 - \frac{m^2}{3z_h}(z-z_h) + \frac{m^4+6m^2-z_h^4\phi'^2(z_h)}{36z_h^2}(z-z_h)^2\right] . \tag{4.80}$$

We now proceed to match the near horizon expression of the fields with the asymptotic solution of these field at any arbitrary point between the horizon and the boundary, say $z = \frac{z_h}{2}$. We would like to make a comment at this point. In the investigation carried out in [37], we matched the near horizon and the asymptotic solutions at any arbitrary point between the horizon and the boundary, that is at $z = \frac{z_h}{\lambda}$ with λ lying between $[1, \infty]$. We observed that the agreement between the matching method values and the numerical values was very good for $\lambda = 3$. However, there is no way to determine a priori a proper matching point. One can in principle choose other matching points also. Thus, the matching method is technically ambiguous compared to the Sturm-Liouville eigenvalue method. In this paper, we choose $\lambda = 2$ for simplicity.
The matching conditions are

$$\phi_h\left(\frac{z_h}{2}\right) = \phi_b\left(\frac{z_h}{2}\right) \quad ; \quad \phi'_h\left(\frac{z_h}{2}\right) = \phi'_b\left(\frac{z_h}{2}\right) \tag{4.81}$$

$$\psi_h\left(\frac{z_h}{2}\right) = \psi_b\left(\frac{z_h}{2}\right) \quad ; \quad \psi'_h\left(\frac{z_h}{2}\right) = \psi'_b\left(\frac{z_h}{2}\right). \tag{4.82}$$

From eq.(4.81), we obtain the following relations

$$\psi^2(z_h) = -\frac{4}{[1-\frac{3b}{2}z_h^4\phi'^2(z_h)]}\left(\frac{\mu}{z_h\phi'(z_h)} + 1 + \frac{3b}{4}z_h^4\phi'^2(z_h)\right) \tag{4.83}$$

$$\rho = -\phi'(z_h)\left[1 + \frac{\psi^2(z_h)}{3}\right] + b\phi'^3(z_h)z_h^4\left[\frac{\psi^2(z_h)}{2} - 1\right] . \tag{4.84}$$

From eq.(4.82), we get

$$\psi_{-/+} = \frac{m^2+12}{6+3\Delta}.\frac{2^{\Delta-1}}{z_h^{\Delta}}.\psi(z_h) \tag{4.85}$$

$$\phi'^2(z_h) = \frac{1}{z_h^4}\left[\frac{144\Delta + 6m^2(6+5\Delta) + m^4(2+\Delta)}{2+\Delta}\right] \Rightarrow \phi'(z_h) = -\frac{\chi(m,\Delta)}{z_h^2} \tag{4.86}$$

where

$$\chi(m,\Delta) = \sqrt{\frac{144\Delta + 6m^2(6+5\Delta) + m^4(2+\Delta)}{2+\Delta}} . \tag{4.87}$$

We shall set $m^2 = -2$ in the rest of our analysis. In eq.(4.85), ψ_- is for $\Delta = \Delta_- = 1$ and ψ_+ is for $\Delta = \Delta_+ = 2$.

We now substitute $\phi'(z_h)$ from eq.(4.86) in eq.(s)(4.83),(4.84) to obtain upto first order in b

$$\psi(z_h) = 2\sqrt{\left(1+\frac{3b}{2}\chi^2\right)\frac{\mu z_h}{\chi} - \left(1+\frac{9b}{4}\chi^2\right)} \quad (4.88)$$

$$= \sqrt{3}\sqrt{\left(1+\frac{3b}{2}\chi^2\right)\frac{\rho z_h^2}{\chi} - \left(1+\frac{5b}{2}\chi^2\right)} \quad (4.89)$$

$$\mu = \frac{\chi}{z_h}\left[\frac{3\rho z_h^2}{4\chi} + \frac{1}{4}\right]. \quad (4.90)$$

The condensation operator and the critical temperature in terms of the chemical potential and the charge density can now be obtained using eq.(s)(4.83)-(4.86) and $T = \frac{3}{4\pi z_h}$:

$$\langle \mathcal{O} \rangle = \gamma_{(\mu)} T_c^{\Delta}\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = \xi_{(\mu)}\mu \quad (4.91)$$

$$\langle \mathcal{O} \rangle = \gamma_{(\rho)} T_c^{\Delta}\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = \xi_{(\rho)}\sqrt{\rho} \quad (4.92)$$

where

$$\gamma_{(\mu)} = \frac{(m^2+12)2^{\Delta}\sqrt{1+\frac{9b}{4}\chi^2}}{(6+3\Delta)}\left(\frac{4\pi}{3}\right)^{\Delta} \quad ; \quad \xi_{(\rho)} = \frac{1+\frac{3b}{2}\chi^2}{1+\frac{9b}{4}\chi^2}\frac{3}{4\pi\chi} \quad (4.93)$$

$$\gamma_{(\rho)} = \frac{(m^2+12)2^{\Delta-1}\sqrt{6(1+\frac{5b}{2}\chi^2)}}{(6+3\Delta)}\left(\frac{4\pi}{3}\right)^{\Delta} \quad ; \quad \xi_{(\rho)} = \sqrt{\frac{1+\frac{3b}{2}\chi^2}{1+\frac{5b}{2}\chi^2}\frac{3}{4\pi\sqrt{\chi}}} \quad (4.94)$$

We now consider $\psi_+ = \langle \mathcal{O} \rangle$, $\psi_- = 0$ which implies that the condensate ψ_+ forms in the absence of the source term ψ_-. Hence the condensation operator and the critical temperature in terms of the chemical potential and the charge density read

$$\langle \mathcal{O}_+ \rangle_\mu = \frac{160\pi^2\sqrt{1+63b}}{27} T_c^2\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.0451\left(\frac{1+42b}{1+63b}\right)\mu \quad (4.95)$$

$$\langle \mathcal{O}_+ \rangle_\rho = \frac{80\pi^2\sqrt{6(1+70b)}}{27} T_c^2\left(1-\frac{T}{T_c}\right)^{1/2} \quad ; \quad T_c = 0.104\sqrt{\frac{1+42b}{1+70b}}\sqrt{\rho} \quad (4.96)$$

In Table 4.4, we have presented the analytical values of the coefficients of the μ (in the $T_c - \mu$ relation) for different values of b obtained from the matching method.

4.4.2 Non-linear effects on holographic free energy

In this section we shall calculate the free energy at a finite temperature of the field theory living on the boundary of the $3+1$- bulk theory. The first step is to consider the action for the action for the Abelian-Higgs sector

$$S_M = \int d^4x\sqrt{-g}\left[\frac{1}{b}\left(1-\sqrt{1+\frac{b}{2}F^{\mu\nu}F_{\mu\nu}}\right) - (D_\mu\psi)^* D^\mu\psi - m^2\psi^*\psi\right]. \quad (4.97)$$

Setting $m^2 = -2$ and expanding the above action keeping terms upto linear order in b yields

$$S_M = \int d^4x \left[\frac{\phi'^2(z)}{2} - \frac{F(z)\psi'^2(z)}{z^2} + \frac{\phi^2(z)\psi^2(z)}{z^2 F(z)} + \frac{2\psi^2(z)}{z^4} + b\frac{z^4\phi'^4(z)}{8} + \mathcal{O}(b^2)\right]. \quad (4.98)$$

Applying the boundary condition ($\phi(z_h) = 0$) and the equations of motion (4.73),(4.74) the on-shell value of the action S_E reads

$$\begin{aligned} S_o &= \int d^3x \left[-\frac{1}{2}\phi(z)\phi'(z)\mid_{z=0} + \frac{F(z)\psi(z)\psi'(z)}{z^2}\mid_{z=0}\right. \\ &\left. - \int_0^{z_h} dz \left(\frac{\phi^2(z)\psi^2(z)}{z^2F(z)} - \frac{3b}{4}\frac{z^2\psi^2(z)\phi^2(z)\phi'^2(z)}{F(z)} - \frac{b}{2}z^3\phi(z)\phi'^3(z)\right)\right] \end{aligned} \quad (4.99)$$

The asymptotic behaviour of $\phi(z)$ and $\psi(z)$ is now substituted in the above expression. This gives

$$\begin{aligned} S_o &= \int d^3x \left[\frac{\mu\rho}{2} + 3\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)\mid_{z=0}\right. \\ &\left. - \int_0^{z_h} dz \left(\frac{\phi^2(z)\psi^2(z)}{z^2(1-z^3/z_h^3)} - \frac{3b}{4}\frac{z^2\psi^2(z)\phi^2(z)\phi'^2(z)}{1-z^3/z_h^3} - \frac{b}{2}z^3\phi(z)\phi'^3(z)\right)\right]. \end{aligned} \quad (4.100)$$

From the above form of the on-shell action, we observe that the term $\left(\frac{\psi_-^2}{z}\right)\mid_{z=0}$ diverges. We study the holographic free energy for the boundary condition $\psi_+ = \langle\mathcal{O}\rangle$, $\psi_- = 0$. To cancel this divergence we add a counter term at the boundary. The counter action which gives the counter term required to cancel this divergence reads

$$S_c = -\int d^3x \left(\sqrt{-h}\psi^2(z)\right)\mid_{z=0} \quad (4.101)$$

where h is the determinant of the induced metric on the AdS boundary. We now evaluate this using the asymptotic behaviour of $\psi(z)$ to obtain

$$S_c = -\int d^3x \left[2\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)\mid_{z=0}\right]. \quad (4.102)$$

Adding S_o and S_c leads to The free energy of the $2+1$-boundary field theory.

$$\Omega = -T(S_o + S_c) = \beta T V_2 \left[-\frac{\mu\rho}{2} - \psi_+\psi_- + I\right] \quad (4.103)$$

where $\int d^3x = \beta V_2$, V_2 being the volume of the $2-$dimensional space of the boundary and the integral I reads upto first order in the BI parameter b

$$\begin{aligned} I &= \int_0^{z_h} dz \frac{\phi^2(z)\psi^2(z)}{z^2(1-z^3/z_h^3)} - \frac{3b}{4}\int_0^{z_h} dz \frac{z^2\psi^2(z)\phi^2(z)\phi'^2(z)}{1-z^3/z_h^3} - b\int_0^{z_h} dz z^3\phi'^3(z)\left[\phi(z) + \frac{z\phi'(z)}{8}\right] \\ &= \int_0^{z_h/2} dz \frac{\phi_b^2(z)\psi_b^2(z)}{z^2(1-z^3/z_h^3)} + \int_{z_h/2}^{z_h} dz \frac{\phi_h^2(z)\psi_h^2(z)}{z^2(1-z^3/z_h^3)} - \frac{3b}{4}\int_0^{z_h/2} dz \frac{z^2\psi_b^2(z)\phi_b^2(z)\phi_b'^2(z)}{1-z^3/z_h^3} \\ &- \frac{3b}{4}\int_{z_h/2}^{z_h} dz \frac{z^2\psi_h^2(z)\phi_h^2(z)\phi_h'^2(z)}{1-z^3/z_h^3} - \frac{b}{2}\int_0^{z_h/2} dz z^3\phi_b(z)\phi_b'^3(z) - \frac{b}{2}\int_{z_h/2}^{z_h} dz z^3\phi_h(z)\phi_h'^3(z). \end{aligned} \quad (4.104)$$

Following the technique in [37] to evaluate the above integral and substituting the value of ρ in terms of μ from eq.(4.90) and using $z_h = \frac{3}{4\pi T}$, we obtain the expression for the free energy in terms of the chemical potential

$$\begin{aligned}\frac{\Omega}{V_2} &= -\frac{\mu\rho}{2} + I \\ &= E_1 T^3 + E_2 T^2\mu + E_3 T\mu^2 + E_4\mu^3 + b\left[E_5 T^3 + E_6 T^2\mu + E_7 T\mu^2 + E_8\mu^3 + E_9\frac{\mu^4}{T} + E_{10}\frac{\mu^5}{T^2}\right]\end{aligned} \tag{4.105}$$

where E_i $(i = 1, 2, ..., 10)$ are constants.
Substituting the value of μ in terms of ρ from eq.(4.90), we obtain the expression for the free energy in terms of the charge density

$$\frac{\Omega}{V_2} = F_1 T^3 + F_2\rho T + F_3\frac{\rho^2}{T} + F_4\frac{\rho^3}{T^3} + b\left[F_5 T^3 + F_6\rho T + F_7\frac{\rho^2}{T} + F_8\frac{\rho^3}{T^3} + F_9\frac{\rho^4}{T^5} + F_{10}\frac{\rho^5}{T^7}\right] \tag{4.106}$$

where F_i $(i = 1, 2, ..., 10)$ are constants.
In Table 4.3, we have shown the values of these constants for the boundary condition $\psi_+ = \langle\mathcal{O}\rangle,\ \psi_- = 0$.

Table 4.3: The values of $E_i,\ F_i,\ (i = 1, 2, ..., 10)$ for $\Delta = \Delta_+ = 2$.

E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
-167.89	8.407	-2.882	0.0338	-10154.2	-587.24	-18.989	1.434	-0.0030	0.000089
F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
-204.04	-3.655	-.0744	.00019	-13745	-118.89	.1683	.0085	.0000054	-1.6×10^{-8}

4.4.3 Thermodynamic geometry

In this section we move on to investigate the thermodynamic geometry of the holographic superconductor. The thermodynamic metric is defined as [154, 155, 156][2]

$$g_{ij} = -\frac{1}{T}\frac{\partial^2\omega(T,\mu)}{\partial x^i\partial x^j} \tag{4.107}$$

[2]Note that the on-shell action (4.99) in our analysis with proper counter term added to it is identified (upto a volume factor) with the free energy $\omega(T,\mu)$ in the grand canonical ensemble. Hence, $\omega(T,\rho)$ should be understood as $\omega(T,\rho(\mu))$ with $x^i = (T,\mu)$. However, the formula $g_{ij} = -\frac{1}{T}\frac{\partial^2\omega(T,\rho)}{\partial x^i\partial x^j}$ with $x^i = (T,\rho)$ can also be used to compute the thermodynamic geometry and holds in the canonical ensemble.

where $\omega = \frac{\Omega}{V_2}$, $x^1 = T$ and $x^2 = \mu$. Hence the components of the metric in terms of μ upto first order in b read

$$\begin{aligned} g_{TT} &= -\left[6E_1 + 2E_2\frac{\mu}{T} + b\left(6E_5 + 2E_6\frac{\mu}{T} + 2E_9\frac{\mu^4}{T^4} + 6E_{10}\frac{\mu^5}{T^5}\right)\right] \\ g_{T\mu} &= g_{\mu T} = -\left[2E_2 + 2F_3\frac{\mu}{T} + b\left(2E_6 + 2E_7\frac{\mu}{T} - 4E_9\frac{\mu^3}{T^3} - 10E_{10}\frac{\mu^4}{T^4}\right)\right] \\ g_{\mu\mu} &= -\left[2E_3 + 6E_4\frac{\mu}{T} + b\left(2E_7 + 6E_8\frac{\mu}{T} + 12E_9\frac{\mu^2}{T^2} + 20E_{10}\frac{\mu^3}{T^3}\right)\right] . \quad (4.108) \end{aligned}$$

The scalar curvature of a general metric

$$ds^2_{th} = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 \quad (4.109)$$

is given by [156]

$$R = \frac{-1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^1}\right)\right] . \quad (4.110)$$

A singularity in R can be found by checking whether the denominator of the right hand side of eq.(4.110) vanishes. The condition of the divergence of R is $det g_{ij} = 0$ which reads

$$g_{TT}g_{\mu\mu} - g^2_{T\mu} = 0 \ . \quad (4.111)$$

The temperature for which the scalar curvature diverges can be obtained by solving this equation keeping terms upto first order in b. This temperature is said to be the critical temperature in the formalism of thermodynamic geometry. We obtain this critical temperature for the case $\psi_+ \neq 0$, $\psi_- = 0$ for different values of b and compare them with the results which have been obtained from the matching method.
Table 4.4 gives the results for $\Delta = \Delta_+ = 2$ obtained from the matching method and the thermodynamic geometry for a set of values of the Born-Infeld parameter b.

Table 4.4: The critical temperature $T_c = \xi_{(\mu)}\mu$ for $\Delta = \Delta_+ = 2$.

b	$\xi_{(\mu)}$ from Matching Method	$\xi_{(\mu)}$ from divergence of R
0.0	0.0451	0.0838
0.01	0.0393	0.0823
0.02	0.0367	0.0815
0.03	0.0329	0.0811

4.5 Conclusions

In this chapter, we have first discussed about the geometrical method of thermodynamics which is the association of a geometrical structure with thermodynamic systems in equilibrium. It was shown that one can get a Riemannian metric with a Euclidean signature from the equilibrium state of a thermodynamic system. The Riemannian scalar curvature can then be computed and captures the details of interactions of the thermodynamic system. It turns out that this framework based on a geometrical structure gives a handle to study critical phenomena. This approach is widely used to study phase transition in ordinary thermodynamical system as well as in gravitational system. Using this framework, a large number of black hole phase transition has been studied. In holographic superconductor models, we are always using a second order phase transition in black hole spacetime to describe the phase transition in boundary theory (i.e. phase transition in superconductors). It is quite natural to investigate whether this thermodynamic geometry approach is effective to study critical phenomena of all holographic superconductor models. For this reason, we consider a holographic superconductor model in presence of Maxwell electrodynamics and another model in presence of Born-Infeld electrodynamics.

We have investigated the critical phenomena for holographic superconductor using this thermodynamic geometry approach. First, we consider Maxwell electrodynamics for a holographic superconductor living in a $2 + 1$-dimensions [36]. We obtain the value of the critical temperature and the condensation operator for two different sets of boundary conditions of the condensation operator using the formalism of thermodynamic geometry. The results are compared with those obtained from the matching method. The matching point is taken to be anywhere between the horizon and the boundary and the results are obtained for a set of values of the matching point. The near horizon expressions obtained by the matching method plays a crucial role in obtaining the free energy of the holographic superconductor. This in turn is used to compute the thermodynamic geometry. It is observed that the results for the critical temperature (in terms of the charge density) from the two approaches, namely, the thermodynamic geometry approach and the matching method are exactly equal with the numerical value $T_c = 0.118\sqrt{\rho}$ when the matching point for the near horizon and the boundary behaviour of the fields is taken to be the $z = z_h/3$ between the horizon and the boundary for the $\psi_+ \neq 0,\ \psi_- = 0$ case. In the next part, we find the critical point of a holographic superconductor coupled to Born-Infeld electrodynamics living in a $2 + 1$-dimensions using the formalism of thermodynamic geometry and matching method [37]. The results from the matching method show that the critical temperature decreases with increase in the Born-Infeld parameter. We observe that the thermodynamic geometry gets affected due to the Born-Infeld parameter which in turn makes the critical temperature to depend on this parameter. The computation of the free energy of the holographic superconductor is carried out in the BI framework and is observed to capture the effects of non-linearity. This then leads to the thermodynamic geometry. Here also we observe that there is a decrease in the critical temperature when the BI parameter b is increased. This tells that the condensation gets harder to form with increase of non-linearity.

It is observed that this geometrical approach is not fully independent method to study critical behavior of holographic superconductors. In this geometrical approach, we have to calculate first the free energy of our models which gives the thermodynamic metric of this geometry. To get free energy of holographic superconductor models in terms of temperature and chemical potential (or charge density), we need to know the relation between the chemical potential and the charge density which is obtained from matching method technique. This implies that thermogeometric approach depends on the matching method technique which is a self-sufficient method to study criticality of holographic superconductors. It is also observed that this geometrical approach is quite good for linear electrodynamics. For non-linear electrodynamics, this approach is not effective because the results obtained from this approach are not in very good agreement with the results obtained from matching method (see Table 4.4) or Sturm-Liouville eigenvalue method. Despite this difference, the geometrical approach is very useful to study the criticality of holographic superconductors. There is an ambiguity to choose the matching point between horizon and boundary. Comparing results from both approaches, namely, the matching method and the geometrical approach, we can argue that the matching point should be taken $z = \frac{z_h}{3}$ in which results match from different methods (geometrical approach, matching method, SL-method, numerical method).

Chapter 5

Conductivity in presence of Born-Infeld electrodynamics

We have investigated the critical temperature and the condensation value in our holographic superconductor models by three different analytic methods in the previous two chapters. In this chapter, we compute the conductivity of holographic superconductors in the framework of BI electrodynamics which has so far been missing in the literature. As mentioned earlier, the importance of this work is to study the effect of non-linear electrodynamics on holographic superconductors. Further, the choice of BI electrodynamics has been made since this is the only non-linear theory of electrodynamics which enjoys the duality symmetry. Here, we proceed to investigate the effects of BI electrodynamics on the conductivity of these systems analytically [38]. The computation of the conductivity analytically incorporating the non-linear effects of BI electrodynamics is important in its own right as it would give information about the dependence of the band gap energy on the non-linear effects coming from BI electrodynamics. We incorporate the effects of the BI parameter in the spacetime metric and also take backreaction into account. We have then calculated the band gap energy from the conductivity expression. We have shown that the band gap energy increases with increase in parameter b. The infinite DC conductivity of our holographic superconductor model has been shown from the AC conductivity expression which has been calculated using two different methods. Before going to the details of computation of conductivity of holographic superconductor, we would like to start with a brief discussion on the Drude Model which is a classical model of conductors.

5.1 Drude Model for DC conductivity

DC conductivity is one of the most important property for the classification of materials. One of the basic properties of superconductor is that the DC conductivity is infinite below a certain critical temperature. Drude model [107] had successfully described the DC conductivity using the following equation

$$m\frac{d\vec{v}}{dt} = e\vec{E} - m\frac{\vec{v}}{\tau} \tag{5.1}$$

where $\vec{v}, \tau, \vec{E}$ are the velocity of electron with mass m and charge e, relaxation time during scattering and electric field respectively. The current in a conductor is given by

$$\vec{J} = ne\vec{v} \tag{5.2}$$

where n is number of electrons. To know the frequency dependency in conductivity (AC conductivity), we consider the external electric field as $\vec{E}(t) = E_x e^{-i\omega t}\hat{i}$. The conductivity along x-direction is then given by

$$\sigma(\omega) = \frac{J_x}{E_x e^{-i\omega t}} = \frac{nev_x(t)}{E_x e^{-i\omega t}} \tag{5.3}$$

where v_x is to be calculated from eq.(5.1) using the external electric field. Substituting $\vec{E}(t) = E_x e^{-i\omega t}\hat{i}$ in eq.(5.1), we find

$$\frac{dv_x(t)}{dt} + \frac{v_x(t)}{\tau} = \frac{eE_x}{m}e^{-i\omega t} \qquad \Rightarrow \qquad v_x(t) = \frac{eE_x\tau}{m}\frac{e^{-i\omega t}}{1 - i\omega\tau}\ . \tag{5.4}$$

Using the above solution (5.4) and definition of conductivity, we get the AC conductivity as

$$\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1 - i\omega\tau} = Re[\sigma(\omega)] + iIm[\sigma(\omega)] \tag{5.5}$$

where $Re[\sigma(\omega)] = \frac{ne^2}{m}\frac{\tau}{1+\omega^2\tau^2}$ and $Im[\sigma(\omega)] = \frac{ne^2}{m}\frac{\omega\tau^2}{1+\omega^2\tau^2}$. For superconductor, the relaxation time $\tau \to \infty$ since there is no scattering. So $Re[\sigma(\omega)] \propto \delta(\omega)$ and $Im[\sigma(\omega)] \propto \frac{1}{\omega}$ for superconductors. This result is expected since the Kramer-Kronig relation connects the real part and the imaginary part by the following relation [159, 160]

$$Im[\sigma(\omega)] = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \frac{Re[\sigma(\omega')]}{\omega' - \omega}\ . \tag{5.6}$$

From this formula, one can conclude that the real part of $\sigma(\omega)$ contains a delta function iff imaginary part has a pole at ω. The DC conductivity of superconductor is infinite since $Re[\sigma(\omega = 0)] \propto \delta(0)$. This model predicts several properties like the Hall effect, electric and thermal frequency with great accuracy. This classical model explains the conductivity of materials in details although it fails to describe the different temperature dependency of conductivity in different type of materials. The main outcome of this classical model which wil be used in the next section, is that the infinite DC conductivity is captured by the imaginary part of AC conductivity.

5.2 Basic set-up and boundary conditions

In this section, we set up the basic formalism and notations which shall be required for subsequent discussion. In $3+1$-dimensions, the action for the model of a holographic superconductor in the framework of Born-Infeld electrodynamics consists of a complex scalar field coupled to a $U(1)$ gauge field in anti-de Sitter black hole spacetime

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R-2\Lambda) + \mathcal{L}_{BI} - (D_\mu\psi)^* D^\mu\psi - m^2\psi^*\psi\right] \tag{5.7}$$

where $\mathcal{L}_{BI} = \frac{1}{b}\left(1-\sqrt{1+\frac{b}{2}F^{\alpha\beta}F_{\alpha\beta} - \frac{b^2}{16}(G^{\alpha\beta}F_{\alpha\beta})^2}\right)$. The last term in $\mathcal{L}_{BI}$ does not contribute in the calculation of conductivity of holographic superconductors also. We know that the plane-symmetric black hole metric takes the form in the presence of backreaction

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2+dy^2)\ . \tag{5.8}$$

Making same ansatz for the gauge field and the scalar field as

$$A_\mu = (\phi(r), 0, 0, 0)\ ,\ \ \psi = \psi(r) \tag{5.9}$$

leads to the following equations of motion for the metric, the gauge and matter fields

$$\begin{aligned} & f'(r) + \frac{f(r)}{r} - \frac{3r}{L^2} + \kappa^2 r \\ \times & \left[f(r)\psi'(r)^2 + \frac{q^2\phi^2(r)\psi^2(r)e^{\chi(r)}}{f(r)} + m^2\psi^2(r) + \frac{1}{b}\left((1-be^{\chi(r)}\phi'(r)^2)^{-\frac{1}{2}} - 1\right)\right] = 0 \end{aligned} \tag{5.10}$$

$$\chi'(r) + 2\kappa^2 r\left(\psi'(r)^2 + \frac{q^2\phi^2(r)\psi^2(r)e^{\chi(r)}}{f(r)^2}\right) = 0 \tag{5.11}$$

$$\phi''(r) + \left(\frac{2}{r} + \frac{\chi'(r)}{2}\right)\phi'(r) - \frac{2}{r}be^{\chi(r)}\phi'(r)^3 - \frac{2q^2\phi(r)\psi^2(r)}{f(r)}(1-be^{\chi(r)}\phi'(r)^2)^{\frac{3}{2}} = 0 \tag{5.12}$$

$$\psi''(r) + \left(\frac{2}{r} - \frac{\chi'(r)}{2} + \frac{f'(r)}{f(r)}\right)\psi'(r) + \left(\frac{q^2\phi^2(r)e^{\chi(r)}}{f(r)^2} - \frac{m^2}{f(r)}\right)\psi(r) = 0 \tag{5.13}$$

where prime denotes derivative with respect to r. We can set $q = 1$ and $L = 1$ without any loss of generality. The conditions $\phi(r_+) = 0$ and $\psi(r_+)$ to be finite imposes the regularity of the fields at the horizon.
The fields near the boundary of the bulk obey [74]

$$\phi(r) = \mu - \frac{\rho}{r} \tag{5.14}$$

$$\psi(r) = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}} \tag{5.15}$$

where

$$\Delta_\pm = \frac{3\pm\sqrt{9+4m^2L^2}}{2} \tag{5.16}$$

are the conformal weights of the conformal field theory living on the boundary. For the choice $\psi_+ = 0$, ψ_- is interpreted as the dual of the expectation value of the condensation operator $\mathcal{O}_\Delta$ in the boundary.
Under changing the coordinate from r to $z = \frac{r_+}{r}$, the field eq.(s) (5.10)-(5.13) look like

$$f'(z) - \frac{f(z)}{z} + \frac{3r_+^2}{z^3} - \frac{\kappa^2 r_+^2}{z^3} \times \left[\frac{z^4}{r_+^2} f(z)\psi'(z)^2 + \frac{\phi^2(z)\psi^2(z)e^{\chi(z)}}{f(z)} + m^2\psi^2(z) + \frac{1}{b}\left((1 - \frac{bz^4}{r_+^2}e^{\chi(z)}\phi'(z)^2)^{-\frac{1}{2}} - 1\right)\right] = 0 \tag{5.17}$$

$$\chi'(z) - \frac{2\kappa^2 r_+^2}{z^3}\left(\frac{z^4}{r_+^2}\psi'(z)^2 + \frac{\phi^2(z)\psi^2(z)e^{\chi(z)}}{f(z)^2}\right) = 0 \tag{5.18}$$

$$\phi''(z) + \frac{\chi'(z)}{2}\phi'(z) + \frac{2}{r_+^2}be^{\chi(z)}\phi'(z)^3 z^3 - \frac{2r_+^2\phi(z)\psi^2(z)}{f(z)z^4}\left(1 - \frac{bz^4 e^{\chi(z)}}{r_+^2}\phi'(z)^2\right)^{\frac{3}{2}} = 0 \tag{5.19}$$

$$\psi''(z) + \left(\frac{f'(z)}{f(z)} - \frac{\chi'(z)}{2}\right)\psi'(z) + \frac{r_+^2}{z^4}\left(\frac{\phi^2(z)e^{\chi(z)}}{f(z)^2} - \frac{m^2}{f(z)}\right)\psi(z) = 0 \tag{5.20}$$

where prime denotes derivative with respect to z. The regularity condition $\phi(r_+)$ becomes $\phi(z = 1) = 0$. In the rest of our work, we set $m^2 = -2$. This leads to two possible values of Δ from eq.(5.16), namely, $\Delta_+ = 2$ and $\Delta_- = 1$.

5.3 Backreacted metric with BI parameter

We now proceed to solve the equation for the metric (5.17) taking into account the effect of the backreaction and the BI parameter b. At $T = T_c$, the matter field vanishes, that is $\psi(z) = 0$. Hence eq.(5.18) reduces to

$$\chi'(z) = 0 \quad \Rightarrow \chi(z) = constant\ . \tag{5.21}$$

Now near the boundary of the bulk, we can set $e^{-\chi(r\to\infty)} \to 1$, i.e. $\chi(r \to \infty) = 0$ which in turn implies $\chi(z = 0) = 0$. This yields $\chi(z) = 0$ from eq.(5.21). Therefore, the gauge field equation (5.19) reduces to

$$\phi''(z) + \frac{2bz^3}{r_{+(c)}^2}\phi'(z)^3 = 0 \tag{5.22}$$

where $r_{+(c)}$ is the horizon radius at $T = T_c$. The solution of this equation for $\phi(z)$ upto $\mathcal{O}(b)$ subject to the boundary condition (5.14) reads [129],[35]

$$\phi(z) = \lambda r_{+(c)}\left\{(1 - z) - \frac{b\lambda^2}{10}(1 - z^5)\right\} \tag{5.23}$$

where

$$\lambda = \frac{\rho}{r_{+(c)}^2}. \tag{5.24}$$

With these solutions in hand, we now proceed to solve the equation for the metric. The metric equation keeping terms upto first order in the Born-Infeld parameter now reads

$$f'(z) - \frac{f(z)}{z} + \frac{3r_{+(c)}^2}{z^3} - \kappa^2\left(\frac{1}{2}\phi'^2(z)z + \frac{3bz^5}{8r_{+(c)}^2}\phi'^4(z)\right) = 0\ . \tag{5.25}$$

Substituting the solution of $\phi(z)$ in the above equation, we obtain the metric equation upto $\mathcal{O}(b)$

$$f' - \frac{f(z)}{z} + \frac{3r_{+(c)}^2}{z^3} - \frac{r_{+(c)}^2\kappa^2\lambda^2}{2}\left(z - \frac{b}{4}\lambda^2 z^5\right) = 0\ . \tag{5.26}$$

Solving this equation and imposing the condition $f(z=1)=0$ to determine the integration constant yields

$$f(z) = \frac{r_{+(c)}}{z^2}g(z) \equiv \frac{r_{+(c)}}{z^2}\left[g_0(z) + g_1(z)\right] \tag{5.27}$$

where

$$g_0(z) = 1 - z^3 \quad ; \quad g_1(z) = \frac{\kappa^2\lambda^2}{2}\left\{z^4 - z^3 - \frac{b\lambda^2}{20}(z^8 - z^3)\right\}\ . \tag{5.28}$$

The above form of the metric includes the effects of backreaction as well as the BI electrodynamics upto first order in the BI parameter b.

5.4 The critical temperature

The Hawking temperature of this black hole spacetime reads

$$T = \frac{f'(r_+)}{4\pi} = -\frac{f'(z=1)}{4\pi r_+} = \frac{3r_+}{4\pi}\left[1 - \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{b\lambda^2}{4}\right)\right] \tag{5.29}$$

which is interpreted as the temperature of the dual field theory at the boundary. Substituting eq.(5.24) in the above equation, we obtain the relation between the critical temperature and the charge density to be

$$T_c = \frac{3}{4\pi}\left[1 - \frac{\kappa_i^2\lambda_{i-1}^2}{6}\left(1 - \frac{b(\lambda^2|_{b=0})}{4}\right)\right]\sqrt{\frac{\rho}{\lambda}} \equiv \xi\sqrt{\rho}\ . \tag{5.30}$$

Note that $\kappa_i^2\lambda^2 = \kappa_i^2(\lambda^2|_{i-1}) + \mathcal{O}(\kappa^4)$ and $b\lambda^2 = b(\lambda^2|_{b=0}) + \mathcal{O}(b^2)$ in the above equation. The procedure to estimate the values of ξ is as follows [35]. The matter field equation (5.20) is first solved near T_c. At $T \to T_c$ (but $T \neq T_c$), the equation for the matter field (5.20) reduces to

$$\psi''(z) + \left(\frac{g'(z)}{g(z)} - \frac{2}{z}\right)\psi'(z) + \left(\frac{\phi^2(z)}{g^2(z)r_{+(c)}^2} + \frac{2}{g(z)z^2}\right)\psi(z) = 0 \tag{5.31}$$

where $\phi(z)$ now corresponds to the solution (5.23). Near the boundary, we define for $\Delta = 1$ [75]

$$\psi(z) = \frac{\langle \mathcal{O}_1 \rangle}{\sqrt{2} r_{+(c)}} z F(z) \tag{5.32}$$

where $F(z)$ is a trial function with $F(0) = 1, F'(0) = 0$, and $\langle \mathcal{O}_1 \rangle$ is the condensation operator. Substituting this form of $\psi(z)$ in eq.(5.31), we obtain

$$\begin{aligned} F''(z) &+ \left\{\frac{g'(z)}{g(z)}\right\} F'(z) + \left\{\left(\frac{g'(z)}{g(z)} - \frac{2}{z}\right)\frac{1}{z} + \frac{2}{g(z)z^2}\right\} F(z) \\ &+ \frac{\lambda^2}{g^2(z)}\left\{(1-z)^2 - \frac{b(\lambda^2|_{b=0})}{5}(1-z)(1-z^5)\right\} F(z) = 0 \ . \end{aligned} \tag{5.33}$$

Recasting the above equation in the Sturm-Liouville form gives

$$\frac{d}{dz}\left\{p(z)F'(z)\right\} + q(z)F(z) + \lambda^2 r(z)F(z) = 0 \tag{5.34}$$

with

$$\begin{aligned} p(z) &= g(z), \\ q(z) &= g(z)\left\{\left(\frac{g'(z)}{g(z)} - \frac{2}{z}\right)\frac{1}{z} + \frac{2}{g(z)z^2}\right\}, \\ r(z) &= \frac{1}{g(z)}\left\{(1-z)^2 - \frac{b(\lambda^2|_{b=0})}{5}(1-z)(1-z^5)\right\}. \end{aligned} \tag{5.35}$$

With these identifications, one can write down an equation for the eigenvalue λ^2 which minimizes the expression

$$\lambda^2 = \frac{\int_0^1 dz\ \{p(z)[F'(z)]^2 - q(z)[F(z)]^2\}}{\int_0^1 dz\ r(z)[F(z)]^2} \ . \tag{5.36}$$

We may now consider the following trial function for the estimation of λ^2

$$F = F_\alpha(z) \equiv 1 - \alpha z^2 \ . \tag{5.37}$$

This function satisfies the conditions $F(0) = 1$ and $F'(0) = 0$. Substituting eq.(s)(5.35) and (5.37) in eq.(5.36) and setting the backreaction parameter $\kappa = 0$ and the Born-Infeld parameter $b = 0$ yields [75]

$$\lambda_\alpha^2 = \frac{2(3 - 3\alpha + 5\alpha^2)}{(2\sqrt{3}\pi - 6\ln 3) + 4(\sqrt{3}\pi + 3\ln 3 - 9)\alpha + (12\ln 3 - 13)\alpha^2} \ . \tag{5.38}$$

The minimum value of $\lambda_\alpha^2 = 1.2683$ and occurs at $\alpha \approx 0.2389$. This in turn gives the value of ξ. Eq.(5.30) now gives the critical temperature

$$T_c = \frac{3}{4\pi\sqrt{\lambda|_{\tilde{\alpha}=0.2389}}}\sqrt{\rho} \approx 0.2250\sqrt{\rho} \tag{5.39}$$

which is in very good agreement with the numerical result $T_c = 0.225\sqrt{\rho}$ [34].
To include the effects of the Born-Infeld and back reaction parameters b and κ, we proceed as follows. We set different values of b and κ and rerun the above analysis to get the value of λ^2. This in turn gives the relation between the critical temperature and the charge density for different values of κ and b. Setting $b = 0.1$ and $\kappa = 0.1$ in eq.(s)(5.35) and (5.28) and using them in eq.(5.36) along with eq.(5.37), we obtain the value of λ^2 in terms of α to be

$$\lambda_\alpha^2 = \frac{0.498954 - 0.498131\alpha + 0.832002\alpha^2}{0.344004 - 0.0825083\alpha + 0.0141889\alpha^2}. \tag{5.40}$$

The minimum value of $\lambda_\alpha^2 = 1.31478$ and occurs at $\alpha \approx 0.23954$. This in turn gives the value of ξ. Eq.(5.30) now gives the critical temperature

$$T_c \approx 0.2225\sqrt{\rho}\ . \tag{5.41}$$

In Table 5.1, we present the values of λ^2 for different values of κ and b. These results shall be used in the next section to calculate the band gap energy for different values of κ and b.

Table 5.1: The values of λ^2 for different values of κ and b.

λ^2	$\kappa = 0.1$	$\kappa = 0.2$	$\kappa = 0.3$
$b = 0.0$	1.2661	1.2593	1.2481
$b = 0.1$	1.3148	1.3076	1.2956
$b = 0.2$	1.3674	1.3597	1.3471
$b = 0.3$	1.4244	1.4162	1.4028

5.5 Computation of conductivity

In this section, we proceed to study the conductivity as a function of frequency, that is optical conductivity. For simplicity, we look at the conductivity along the x-direction. By the gauge/gravity duality, the fluctuations in the Maxwell field in the bulk gives rise to the conductivity.
Making the ansatz $A_\mu = (0, 0, \varphi(r, t), 0)$ with $\varphi(r, t) = A(r)e^{-i\omega t}$ and neglecting terms of $\mathcal{O}(b^2)$ and $\mathcal{O}(\omega^2 b)$ leads to the following equation of motion for $A(r)$

$$A''(r) + \frac{f'(r)A'(r)}{f(r)}\left\{1 + \frac{b}{r^2}f(r)A'^2(r)e^{-2i\omega t}\right\} - \frac{be^{-2i\omega t}}{2r^2}A'^3(r)\left(f'(r) - \frac{2f(r)}{r}\right) + \left[\frac{\omega^2}{f^2(r)} - \frac{2\psi^2(r)}{f(r)}\left(1 + \frac{3b}{2r^2}f(r)A'^2(r)e^{-2i\omega t}\right)\right]A(r) = 0\ . \tag{5.42}$$

This equation is very difficult to solve analytically. However, in principle we can employ a perturbative approach.
To make progress, we start by neglecting the non-linear terms in the above equation[1]. This reads

$$A''(r) + \frac{f'(r)A'(r)}{f(r)} + \left[\frac{\omega^2}{f^2(r)} - \frac{2\psi^2(r)}{f(r)}\right] A(r) = 0 \ . \tag{5.43}$$

Note that the effect of the BI parameter is contained in the metric. The perturbative technique involves solving this equation and then replacing this solution in the $\mathcal{O}(b)$ terms in eq.(5.42) and solving the equation once again.
We now move to the tortoise coordinate which is defined by

$$\begin{aligned} r_* &= \int \frac{dr}{f(r)} = -\frac{1}{r_+}\int \frac{dz}{g_0(z)+g_1(z)} \approx -\frac{1}{r_+}\left\{\int \frac{dz}{g_0(z)} - \int \frac{g_1(z)}{g_0^2(z)}dz\right\} \\ &= ln(1-z)^{\frac{1}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}} + ln(1+z+z^2)^{-\frac{1}{6r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1+\frac{13}{20}b\lambda^2\right)\right\}} \\ &+ \frac{\kappa^2\lambda^4 b}{120r_+}(1-z^3) - \frac{\kappa^2\lambda^2(z+\frac{b\lambda^2}{20})}{6r_+(1+z+z^2)} - \frac{1}{\sqrt{3}r_+}\left\{1-\frac{\kappa^2\lambda^2}{2}+\frac{\kappa^2\lambda^4 b}{120}\right\} tan^{-1}\frac{\sqrt{3}z}{2+z} \ . \end{aligned} \tag{5.44}$$

The integration constant has been calculated from the condition $r_*(z) = 0$ at $z = 0$. The wave equation (5.43) in tortoise coordinate reads

$$\frac{d^2A}{dr_*^2} + \left[\omega^2 - V\right] A = 0 \tag{5.45}$$

where

$$V = 2\psi^2 f \ . \tag{5.46}$$

We now employ a trick to solve this equation. We first solve this equation for $V = 0$ which implies that we solve only the ω-dependent part of the equation. The solution reads

$$A \sim e^{-i\omega r_*} \sim (1-z)^{-\frac{i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}} \tag{5.47}$$

where we have considered only leading order terms in r_* that is, $r_* = ln(1-z)^{\frac{1}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}}$ in obtaining the above expression.

5.5.1 Method 1: To estimate the band gap energy

We now want to know the function which is independent of ω and has only z dependence. To do this we first write down eq.(5.45) in z-coordinate. This reads

$$g(z)\frac{d^2A(z)}{dz^2} + g'(z)\frac{dA(z)}{dz} + \left[\frac{\omega^2}{r_+^2 g(z)} - \frac{2\psi^2(z)}{z^2}\right] A(z) = 0 \ . \tag{5.48}$$

[1]This is done to carry out the analysis analytically.

We now write $A(z)$ as a product of the ω-dependent part and a function of z which we need to determine. Hence, the gauge field reads

$$A(z) = (1-z)^{-\frac{i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}} G(z) \tag{5.49}$$

where $G(z)$ is regular at the horizon of the black hole.
Substituting this in eq.(5.48), we obtain

$$g(z)G''(z) + \left[\frac{2i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\frac{g(z)}{1-z} + g'(z)\right]G'(z)$$
$$\left[\frac{i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\left\{\frac{i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}+1\right\}\frac{g(z)}{(1-z)^2} +\right.$$
$$\left. + \frac{i\omega}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\frac{g'(z)}{1-z} + \frac{\omega^2}{r_+^2 g(z)} - \frac{2\psi^2(z)}{z^2}\right]G(z) = 0\ . \tag{5.50}$$

For $\Delta = 1$, we know that $\psi(z) = \frac{\langle\mathcal{O}_1\rangle}{\sqrt{2}r_+}F(z)z$ where $F(0) = 1$. For simplification, we consider $F(z)$ to be 1 because we are neglecting order $\mathcal{O}(z^3)$ term. Substituting this in eq.(5.50), we get

$$3g_0(z)G''(z) + \left[\frac{2i\omega C_1}{r_+}(1+z+z^2) - 9C_2(z)z^2\right]G'(z) + \left[\frac{i\omega C_1}{r_+}\left(1+z+z^2-3C_2(z)z^2\right)\right.$$
$$\left.\times\frac{1}{1-z} + \frac{\omega^2}{3r_+^2}\left\{9C_3(z) - (1+z+z^2)^2C_1^2\right\}\frac{1}{1-z^3} - \frac{3\langle\mathcal{O}_1\rangle^2}{r_+^2}C_4(z)\right]G(z) = 0 \tag{5.51}$$

where

$$C_1 = 1 + \frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right) \quad ; \quad C_2(z) = \frac{1+\frac{g_1'(z)}{g_0'(z)}}{1+\frac{g_1(z)}{g_0(z)}}$$
$$C_3(z) = \frac{1}{\left(1+\frac{g_1(z)}{g_0(z)}\right)^2} \quad ; \quad C_4(z) = \frac{1}{1+\frac{g_1(z)}{g_0(z)}}\ . \tag{5.52}$$

Keeping terms upto order z^3 in the above equation yields

$$3g_0(z)G''(z) + \left[\frac{2i\omega C_1}{r_+}(1+z+z^2) - 9C_2(z)z^2\right]G'(z) + \left[\frac{i\omega C_1}{r_+}\left\{1+2z+3z^2(1-C_2)\right.\right.$$
$$\left. +3z^3(1-C_2)\right\} + \frac{\omega^2}{3r_+^2}\left\{\frac{C_5 + (C_5-2C_1^2)z + (C_5-5C_1^2)z^2 + (C_5-7C_1^2)z^3}{1+z+z^2}\right\}$$
$$\left. - \frac{3\langle\mathcal{O}_1\rangle^2}{r_+^2}C_4(z)\right]G(z) = 0 \tag{5.53}$$

where $C_5(z) = 9C_3(z) - C_1^2$. To solve this equation, we rescale it by letting $z = \frac{z'}{a}$, where $a = \frac{<\mathcal{O}_1>}{r_+}$ and then take the $a \to \infty$ limit which corresponds to the low temperature regime [75]. This leads to

$$G''(z') - G(z') = 0\ . \tag{5.54}$$

The solution of this equation reads

$$\begin{aligned} G(z') &= C_+ e^{z'} + C_- e^{-z'} \\ \Rightarrow G(z) &= C_+ e^{az} + C_- e^{-az} \\ &= C_+ e^{\frac{<\mathcal{O}_1>}{r_+}z} + C_- e^{-\frac{<\mathcal{O}_1>}{r_+}z} \ . \end{aligned} \tag{5.55}$$

The information about the integration constants C_+ and C_- can be obtained from the appropriate boundary condition. For $\Delta = 1$, the boundary condition can be obtained from eq.(5.53) by setting $z = 1$ in the equation. This gives

$$G'(1) + \left[\frac{2\langle\mathcal{O}_1\rangle^2 C_4(1)}{r_+(3 - \frac{2i\omega}{r_+}C_1(1))} - \frac{\frac{i\omega}{3r_+}\left\{3C_1(1) - \frac{i\omega}{9r_+}(4C_5(1) - 14C_1^2(1))\right\}}{3 - \frac{2i\omega}{r_+}C_1(1)} \right] G(1) = 0 \tag{5.56}$$

where

$$\begin{aligned} C_2(1) = 1 \quad &; \quad C_3(1) \approx 1 + \frac{\kappa^2\lambda^2}{3}\left(1 - \frac{1}{4}b\lambda^2\right) \\ C_4(1) \approx 1 + \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{1}{4}b\lambda^2\right) \quad &; \quad C_5(1) = 9C_3(1) - C_1^2 \ . \end{aligned} \tag{5.57}$$

Substituting $G(1)$ and $G'(1)$ from eq.(5.55) in the boundary condition (5.56) yields upto first order in ω

$$\frac{C_+}{C_-} = -e^{-2a}\left[\frac{aC_4(1) - 3}{aC_4(1) + 3} + \frac{2iC_1\omega}{ar_+}\frac{(2C_4(1)a^2 - 3)}{(C_4(1)a + 3)^2} + \mathcal{O}(\omega^2)\right] \ . \tag{5.58}$$

We finally obtain the solution for $A(z)$ from eqs.(5.49), (5.55). This reads

$$A(z) = (1 - z)^{-\frac{i\omega}{3r_+}\left\{1 + \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{1}{4}b\lambda^2\right)\right\}}\left[C_+ e^{\frac{\langle\mathcal{O}_1\rangle}{r_+}z} + C_- e^{-\frac{\langle\mathcal{O}_1\rangle}{r_+}z}\right] \ . \tag{5.59}$$

To obtain the conductivity, we now expand the gauge field about $z = 0$:

$$A(z) = A(0) + zA'(0) + \mathcal{O}(z^2) \ . \tag{5.60}$$

Now in general A_x can be written as

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r_+}z + \mathcal{O}(z^2) \ . \tag{5.61}$$

Comparing eq.(5.60) and eq.(5.61), we have

$$A_x^{(0)} = A(0) \quad ; \quad A_x^{(1)} = r_+ A'(0) \ . \tag{5.62}$$

Now from the definition of conductivity and gauge/gravity correspondence, we have

$$\begin{aligned} \sigma(\omega) &= \frac{\langle J_x\rangle}{E_x} = \frac{iA_x^{(1)}}{\omega A_x^{(0)}} = -\frac{i}{\omega}\frac{r_+A'(z=0)}{A(z=0)} \\ &= \frac{i}{\omega}\frac{1 - \frac{C_+}{C_-}}{1 + \frac{C_+}{C_-}}\langle\mathcal{O}_1\rangle + \frac{1}{3}\left\{1 + \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{1}{4}b\lambda^2\right)\right\} \ . \end{aligned} \tag{5.63}$$

Substituting the value of $\frac{C_+}{C_-}$, we obtain the low frequency expression for the conductivity to be

$$\sigma(\omega) = \frac{i\langle\mathcal{O}_1\rangle}{\omega}\left[1 + 2e^{-2a}\frac{aC_4(1)-3}{aC_4(1)+3} + 4e^{-2a}\frac{iC_1\omega}{ar_+}\frac{(2C_4(1)a^2-3)}{(C_4(1)a+3)^2}\right.$$
$$\left. + \frac{1}{3}\left\{1 + \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{1}{4}b\lambda^2\right)\right\}\right] . \qquad (5.64)$$

A few observations are in order now. This expression for the conductivity is valid in the low temperature limit. The above result is useful in estimating the band gap energy of the holographic superconductors [75]. This can be estimated as follows. The DC conductivity is defined as the real part of σ at $\omega = 0$. This reads

$$Re\ \sigma(\omega = 0) \sim e^{-2a}\left[1 + \mathcal{O}(1/a)\right] \approx e^{-2\frac{\langle\mathcal{O}_1\rangle}{r_+}} . \qquad (5.65)$$

Substituting the value of r_+ in terms of the Hawking temperature T from eq.(5.29) in the above equation, we finally obtain

$$Re\ \sigma(\omega = 0) \sim e^{-\frac{E_g}{T}} \qquad (5.66)$$

where

$$E_g = \frac{3}{2\pi}\left\{1 - \frac{\kappa^2\lambda^2}{6}\left(1 - \frac{1}{4}b\lambda^2\right)\right\}\langle\mathcal{O}_1\rangle. \qquad (5.67)$$

E_g is identified to be the band gap energy. Note that the band gap energy gets corrected due to the backreaction and the BI parameter. We observe that the effect of the BI parameter vanishes when $\kappa = 0$. We also recover the band gap energy $E_g = \frac{3\langle\mathcal{O}_1\rangle}{2\pi} \approx 0.48\langle\mathcal{O}_1\rangle$ [75] for $\kappa = 0$. Using the results in Table 5.1, we have calculated the band gap energy for different values of κ and b. These results are displayed in Table 5.2. We recall that the values of λ^2 (appearing in eq.(5.67)) have been estimated using the Sturm-Liouville eigenvalue approach for different values of κ [35]. The results indicate that for a particular value of the BI parameter b, the band gap energy decreases with increasing values of backreaction parameter κ. Further, for a particular value of κ, the band gap energy increases with increasing values of the BI parameter b. It would be nice to compare our analytical results with numerical studies which are presently missing in the literature to the best of our knowledge.

Table 5.2: The values of $\frac{E_g}{\langle\mathcal{O}_1\rangle}$ for different values of κ and b.

$\frac{E_g}{\langle\mathcal{O}_1\rangle}$	$\kappa = 0.1$	$\kappa = 0.2$	$\kappa = 0.3$
$b = 0.0$	0.47646	0.47344	0.46845
$b = 0.1$	0.47649	0.47356	0.46873
$b = 0.2$	0.47652	0.47369	0.46901
$b = 0.3$	0.47655	0.47382	0.46929

5.5.2 Method 2: Self-consistent approach

We now present the self-consistent approach to obtain the conductivity expression. Here we essentially follow the approach in [75]. We first replace the potential with its average $\langle V \rangle$ in a self-consistent way. With this approximation, the solution of eq.(5.45) reads

$$A \sim e^{-i\sqrt{\omega^2 - \langle V \rangle} r_*} \sim (1-z)^{-i\sqrt{\omega^2-\langle V \rangle}\frac{1}{3r_+}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}} . \tag{5.68}$$

Once again from the definition of conductivity and gauge/gravity dictionary, we obtain from eq.(5.63)

$$\begin{aligned} \sigma(\omega) &= -\frac{i}{\omega}\frac{r_+ A'(z=0)}{A(z=0)} \\ &= \frac{1}{3}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\sqrt{1-\frac{\langle V \rangle}{\omega^2}} . \end{aligned} \tag{5.69}$$

We now need to estimate the average value of the potential. This reads

$$\langle V \rangle = \frac{\int_{-\infty}^{0} dr_* V A^2(r_*)}{\int_{-\infty}^{0} dr_* A^2(r_*)} . \tag{5.70}$$

From eq.(5.46) and $\psi(z) = \frac{\langle \mathcal{O}_\Delta \rangle}{\sqrt{2} r_+^\Delta} z^\Delta F(z)$, the potential reads

$$V \approx \frac{\langle \mathcal{O}_\Delta \rangle^2}{r_+^{2\Delta-2}} z^{2\Delta-2}\left[g_0(z) + g_1(z)\right] \tag{5.71}$$

where we consider $F(z) \approx 1$. The main contribution to the average value $\langle V \rangle$ in eq.(5.70) is from the vicinity of the boundary where $r_* \approx -\frac{z}{r_+}$. Substituting eq.(5.71) in eq.(5.70), we obtain the expression for $\langle V \rangle$ to be

$$\begin{aligned} \langle V \rangle &\simeq \frac{\langle \mathcal{O}_\Delta \rangle^2}{r_+^{2\Delta-2}}\left\{\frac{\int_0^\infty dz e^{2i\sqrt{\omega^2-\langle V \rangle}\frac{z}{r_+}} z^{2\Delta-2} g_0(z)}{\int_0^\infty dz e^{2i\sqrt{\omega^2-\langle V \rangle}\frac{z}{r_+}}}\right\} \\ &\simeq \langle \mathcal{O}_\Delta \rangle^2 \left[\frac{\Gamma(2\Delta-1)}{(-2i\sqrt{\omega^2-\langle V \rangle})^{2\Delta-2}}\right] . \end{aligned} \tag{5.72}$$

This is the self consistent equation for the average value of the potential $\langle V \rangle$, which depends on the frequency ω. At the low frequency limit, we set $\omega = 0$ in eq.(5.72) which leads to

$$\langle V \rangle^\Delta = \frac{\langle \mathcal{O}_\Delta \rangle^2}{2^{2\Delta-2}}\Gamma(2\Delta-1) . \tag{5.73}$$

For $\Delta = 1$, this gives

$$\langle V \rangle = \langle \mathcal{O}_1 \rangle^2 . \tag{5.74}$$

Using this in eq.(5.69), the conductivity is given by

$$\begin{aligned}\sigma(\omega) &= \frac{1}{3}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\sqrt{1-\frac{\langle\mathcal{O}_1\rangle^2}{\omega^2}}\\ &= \frac{i\langle\mathcal{O}_1\rangle}{3\omega}\left\{1+\frac{\kappa^2\lambda^2}{6}\left(1-\frac{1}{4}b\lambda^2\right)\right\}\sqrt{1-\frac{\omega^2}{\langle\mathcal{O}_1\rangle^2}}\ .\end{aligned} \tag{5.75}$$

This expression is valid for all temperature limit and it shows that the imaginary part of the conductivity has a pole at $\omega = 0$. Using Kramer-Kronig relation [159, 160] we can say that real part of the conductivity is a delta function. This implies that DC conductivity ($\omega = 0$) of holographic superconductor is infinite. This argument has been also reflected in Drude model of DC conductivity. This feature was also observed in [75]. However, this method does not capture the expression for the band gap energy. It can be observed that the above result agrees in form at the leading order with the result obtained in eq.(5.64).

5.6 Conclusions

In this chapter, we have focused on the analytic computation of conductivity of holographic superconductor in the presence of Born-Infeld electrodynamics [38]. The Drude model of DC conductivity has been discussed in the first section. This classical model tells us that the imaginary part of AC conductivity of a superconductor captures the infinite DC conductivity through Kramer-Kronig relation. In all holographic superconductor models, it was numerically shown that only the imaginary part of conductivity show infinite DC conductivity. In [75], it was also shown analytically that only the imaginary part of conductivity captures infinite DC conductivity. In the next section, we have mentioned the basic set up and required boundary conditions for our analysis. By employing a perturbative approach, we have then computed the backreacted bulk spacetime metric taking into account the effect of the Born-Infeld electrodynamics. Using this backreacted metric, we have calculated the critical temperature and the eigenvalue λ of Sturm-Liouville method for different value of BI parameter which are needed to estimate the band gap energy value. The critical temperature and the eigenvalue λ capture the effects of matter fields (gauge and scalar field) on the spacetime metric. Inspiring the analytical method in [75], we now engage to compute the analytic conductivity of holographic superconductors in the framework of Born-Infeld electrodynamics away from the probe limit. We found the analytic expression for conductivity which is found to contain the effects of the backreaction parameter κ and the BI parameter b. From the real part of the conductivity (computed at $\omega = 0$), the band gap energy is obtained. It is observed that the energy gap gets corrected from the standard value due to the parameters κ and b. The dependence of the band gap energy on the non-linear effects coming from BI electrodynamics is manifest. The results show that the band gap energy decreases with increase in the values of the backreaction parameter κ for a fixed value of the BI parameter b. Moreover, it increases with increase in b for a fixed value of κ. We then perform the computation of conductivity by following a self-consistent approach. From the self-consistent approach, we have shown that DC conductivity of our holographic superconductors is infinite which is consistent with any holographic superconductor model.

Chapter 6

Response of holographic superconductors to magnetic field

6.1 Introduction

The unique property of superconductors is that it expels any external magnetic field below the critical temperature. During the measurement of magnetic field distribution outside the superconducting sample, this unique property was first observed by the German physicist W. Meissner [115]. This phenomena is known as Meissner effect which describes that magnetic field does not penetrate into superconductors completely. If the external magnetic field is high enough, it completely destroys the superconductivity of the material. The magnetic field at which Meissner effect is completely destroyed, is known as the critical magnetic field which depends on the temperature in following way [112]

$$B_c = B_0 \left[1 - \left(\frac{T}{T_c} \right)^2 \right] \tag{6.1}$$

where B_c is the critical temperature and T_c is the critical temperature. Based on the response of external magnetic field, superconductors are divided into two classes, namely, type-I and type-II superconductors. For type-I superconductors, the complete Meissner state is destroyed by the critical magnetic field. For type-II superconductors, there is a state in which both Meissner state and normal state are present, known as vortex state [112]. Below the lower critical magnetic field, only the Meissner state exists and the vortex state exists below the upper critical magnetic field ($100kG$) for type-II superconductors. The graphical representation of Meissner effects are shown in Figure 6.1 :

The high T_c superconductors are type-II superconductors since they show vortex formation in between the lower and the upper critical magnetic field. Above the lower critical temperature, magnetic flux penetrates into superconductor to form vortex since the Gibbs free energy in vortex state is smaller than the Gibbs free energy in Meissner state. The lower critical magnetic field and the upper critical magnetic field are given by [113]

$$B_{c_1} = \frac{\phi_0}{4\pi\lambda^2} \ln \frac{\lambda}{\xi} \quad , \quad B_{c_2} = \frac{\phi_0}{2\pi\xi^2} \tag{6.2}$$

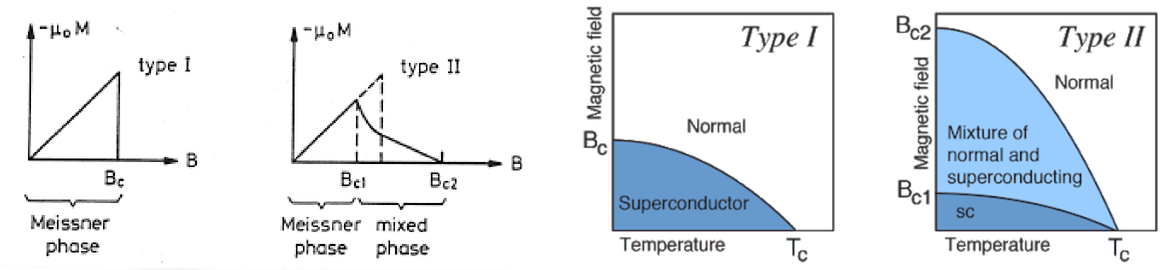

Figure 6.1: Response of Type-I and Type-II superconductors to magnetic field

where $\phi_0 = \frac{h}{2e}$ is quantized magnetic flux, λ and ξ are penetration depth and coherence length respectively. For high T_c superconductors (Cuprate-layer based superconductors), the typical value of penetration depth and coherence length are $120nm$ and $2nm$ respectively [161]. From the above expressions, we find the lower and upper critical magnetic field as $B_{c_1} \sim 0.47kG$ and $B_{c_2} \sim 823kG$. In [162, 163], the non-linear Meissner effect had been studied for cuprate-layer based superconductors which are known as unconventional superconductors. These superconductors are always strongly coupled superconductors and they are type-II superconductors.

Holographic superconductors are the model for these unconventional superconductors which are strongly coupled superconductors. Holographic superconductors are type-II superconductors which have been shown in [164]. Using the same model mentioned in [34], vortex configuration has been investigated and the lower and upper critical magnetic field has been calculated by studying free energy as a function of external magnetic field. The critical magnetic field of the holographic superconductor model [34] was first investigated in [165]. Later, the vortex solution in holographic superconductors has been found, which indicates that holographic superconductors are type-II superconductors. As mentioned in [163], unconventional superconductors shows non-linear Meissner effect. It is obvious to ask whether holographic superconductors show non-linear Meissner effect or not. For holographic superconductor model, we assume that the non-linearity comes from the electrodynamics part. There have been several investigations to understand the Meissner-like effect in holographic superconductors in the presence of Maxwell electrodynamics ([165],[166],[167]) which is linear electrodynamics. In [146], non-linear Meissner effect has been investigated by considering Born electrodynamics. In other non-linear electrodynamics [168],[169] have been considered to study the response of holographic superconductors to an external magnetic field.
However, the study of non-linear effects on the critical magnetic field due to Born-Infeld (BI) electrodynamics [59]-[60] has been not carried out so far in the literature. The difference between the BI electrodynamics and Born electrodynamics is apparent only when the magnetic field is switched on. This is because of the extra $\vec{E}.\vec{B}$ term in the BI theory which is absent in the Born theory. The BI theory proposed by Born and Infeld [59] and analyzed in detail by Dirac [60], was favoured over the Born theory [58] as it was constructed out of two Lorentz invariant quantities $F^{\alpha\beta}F_{\alpha\beta}$ and $F^{\alpha\beta}G_{\alpha\beta}$ thereby leading to a more general theory of nonlinear electrodynamics. Further, it was found that the Born theory exhibited vacuum birefringence whereas

the BI theory does not exhibit vacuum birefringence [170]. Another motivation for looking at the effects of BI electrodynamics on holographic superconductors is to check whether the $\vec{E}.\vec{B}$ term increases or decreases the critical magnetic field compared to the Born electrodynamics. These features provide enough motivation to study holographic superconductors in the presence of BI electrodynamics.
In this chapter, we investigate the effects of magnetic field on holographic superconductors by considering BI electrodynamics [39]. Our intention is to study how the presence of extra $\vec{E}.\vec{B}$ term in BI theory affects the Meissner effect. In particular we would like to observe the non-linear effects coming from BI electrodynamics on the critical magnetic field at which superconducting order gets destroyed. We calculate analytically the critical magnetic field at which the superconducting state becomes normal metallic state. In this work we use the matching method technique in which we match the asymptotic behavior of fields with the horizon behavior of the fields. The critical magnetic field obtained from the BI electrodynamics incorporates the non-linear effects.

6.2 Basic formalism

In $3+1$-dimensions, the action for the model of a holographic superconductor in the framework of BI electrodynamics consists a complex scalar field coupled to a $U(1)$ gauge field in AdS black hole spacetime reads [1]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R-2\Lambda)+\frac{1}{b}\left(1-\sqrt{1+\frac{b}{2}F^{\mu\nu}F_{\mu\nu}-\frac{b^2}{16}(G^{\mu\nu}F_{\mu\nu})^2}\right)\right.$$
$$\left.-(D_\mu\psi)^*D^\mu\psi-m^2\psi^*\psi\right] . \quad (6.3)$$

It should be noted that in the existing literature on holographic superconductors one considers the Born theory [58] instead of the Born-Infeld theory [59], [60]. The Lagrangian density of the Born theory [58] is given by

$$\mathcal{L}_B = \frac{1}{b}\left(1-\sqrt{1+\frac{b}{2}F^{\alpha\beta}F_{\alpha\beta}}\right) . \quad (6.4)$$

Later on Born and Infeld favoured the following Lagrangian density [59]

$$\mathcal{L}_{BI} = \frac{1}{b}\left(1-\sqrt{1+\frac{b}{2}F^{\alpha\beta}F_{\alpha\beta}-\frac{b^2}{16}(G^{\alpha\beta}F_{\alpha\beta})^2}\right) \quad (6.5)$$

over the Born theory given in eq.(6.4). In the BI theory, the Born theory gets augmented by the third term under the square root in eq.(6.5). This term turns out to be very important when we study the effects of the magnetic field on holographic superconductors since this term is proportional to $\vec{E}.\vec{B}$ and would give an additional contribution along with $F_{\mu\nu}F^{\mu\nu}$ for a non-zero magnetic field. However, in the absence of the magnetic field, there is no difference between the Born and the BI

[1] This discussion is based on our work [39].

theories. The equation of motion for the gauge and matter fields read from the above action

$$\partial_\alpha \left[\frac{\sqrt{-g}F^{\alpha\beta}}{\sqrt{1+\frac{b}{2}F^{\mu\nu}F_{\mu\nu}-\frac{b^2}{16}(G^{\mu\nu}F_{\mu\nu})^2}} \right] - \frac{b}{4}\partial_\alpha \left[\frac{\sqrt{-g}G^{\alpha\beta}G^{\mu\nu}F_{\mu\nu}}{\sqrt{1+\frac{b}{2}F^{\mu\nu}F_{\mu\nu}-\frac{b^2}{16}(G^{\mu\nu}F_{\mu\nu})^2}} \right]$$
$$= 2\sqrt{-g}q^2A^\beta|\psi|^2 + iq\sqrt{-g}\left[\psi^*\partial^\beta\psi - \psi\partial^\beta\psi^*\right] \quad (6.6)$$

$$\partial_\alpha\left[\sqrt{-g}\partial^\alpha\psi\right] = \sqrt{-g}\left[q^2A_\mu A^\mu + m^2\right]\psi + iq\left[2\sqrt{-g}A^\mu\partial_\mu\psi + \sqrt{-g}(\partial_\mu A^\mu)\psi\right.$$
$$\left. + (\partial_\mu\sqrt{-g})A^\mu\psi\right] \ . \quad (6.7)$$

To know the response of any external magnetic field on holographic superconductors, we first employ the matching method to calculate the critical temperature which will be used to calculate the critical magnetic field. In the next section we have a brief discussion on the investigation of the critical temperature by matching method.

6.3 The critical temperature

The plane-symmetric black hole geometry reads (setting the AdS radius $L = 1$)

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \quad ; \quad f(r) = r^2\left(1 - \frac{r_+^3}{r^3}\right) \ . \quad (6.8)$$

The Hawking temperature of the black hole spacetime (6.8) takes form as

$$T = \frac{f'(r_+)}{4\pi} = \frac{3r_+}{4\pi} \quad (6.9)$$

which is interpreted as the temperature of the dual field theory at the boundary. Making the same ansatz for the gauge field and the scalar field as in chapter 3, leads to the following equations of motion for the gauge and matter fields

$$\phi''(r) + \frac{2}{r}\phi'(r) - \frac{2}{r}b\phi'(r)^3 - \frac{2q^2\phi(r)\psi^2(r)}{f(r)}(1 - b\phi'(r)^2)^{\frac{3}{2}} = 0 \quad (6.10)$$

$$\psi''(r) + \left(\frac{2}{r} + \frac{f'(r)}{f(r)}\right)\psi'(r) + \left(\frac{q^2\phi^2(r)}{f(r)^2} - \frac{m^2}{f(r)}\right)\psi(r) = 0 \quad (6.11)$$

where prime denotes derivative with respect to r. The conditions $\phi(r_+) = 0$ and $\psi(r_+)$ to be finite imposes the regularity of the fields at the horizon.
Setting $q = 1$ and under changing the coordinate from r to $z = \frac{r_+}{r}$, the field eq.(s) (6.10)-(6.11) look like

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'(z)^3 - \frac{2r_+^2\psi^2(z)}{z^4 f(z)}\left(1 - \frac{bz^4}{r_+^2}\phi'(z)^2\right)^{\frac{3}{2}}\phi(z) = 0 \quad (6.12)$$

$$\psi''(z) + \frac{f'(z)}{f(z)}\psi'(z) + \frac{r_+^2}{z^4}\left(\frac{\phi^2(z)}{f(z)^2} - \frac{m^2}{f(z)}\right)\psi(z) = 0 \quad (6.13)$$

where prime denotes derivative with respect to z. To solve these equations, we have to impose the boundary behavior of the fields. The asymptotic behavior of fields are [74]

$$\phi(z) = \mu - \frac{\rho}{r_+} z \tag{6.14}$$

$$\psi(r) = \frac{J_-}{r_+^{\Delta_-}} z^{\Delta_-} + \frac{J_+}{r_+^{\Delta_+}} z^{\Delta_+} \tag{6.15}$$

where $\Delta_\pm = \frac{3 \pm \sqrt{9+4m^2}}{2}$.
For $m^2 = -2$, we have $\Delta_+ = 2$ and $\Delta_- = 1$. Here, we consider the case $J_+ = 0$, so the relevant conformal dimension is $\Delta = \Delta_- = 1$ and hence the matter field near the AdS boundary is given by

$$\psi(z) = \frac{J_-}{r_+} z \tag{6.16}$$

where J_- is the condensation operator. In order to study the effect of the magnetic field, we first need to investigate the relation between the critical temperature and the charge density. This we do using the matching method technique in which we match the asymptotic behaviour of fields with the horizon behaviour of field at any arbitrary point (z_m) between $[0, 1]$. The details of this study are presented in the Appendix.
The critical temperature T_c at zero magnetic field reads [146]

$$T_c = \frac{3}{4\pi} \frac{\sqrt{\rho}}{\sqrt{\tilde{\beta}\{1 + 2b\tilde{\beta}^2(1 - z_m)\}}} \tag{6.17}$$

where

$$\tilde{\beta} = 2\sqrt{\frac{1 + 2z_m^2}{1 - z_m^2}} \ . \tag{6.18}$$

These results will be used in the subsequent discussion to find the effect of the magnetic field on holographic superconductors.

6.4 Effect of magnetic field

In this section, we add a magnetic field in the bulk. The asymptotic value of this magnetic field represents a magnetic field in the boundary field theory. The following ansatz is taken to study the Meissner effect of holographic superconductors

$$A_\mu = (\phi(r), 0, 0, Bx) \ , \qquad \psi \equiv \psi(r, x) \ . \tag{6.19}$$

Using the above ansatz, we obtain from eq.(s)(6.6,6.7)

$$\left(1+bB^2\right)\partial_r\left[\frac{r^2\phi'(r)}{\sqrt{1+b(\frac{B^2}{r^4}-\phi'^2(r))-b^2B^2\phi'^2(r)}}\right] = \frac{2q^2r^2\psi^2(r,x)}{f(r)}\phi(r) \quad (6.20)$$

$$\partial_r^2\psi(r,x)+\left(\frac{f'(r)}{f(r)}+\frac{2}{r}\right)\partial_r\psi(r,x)-\frac{m^2}{f(r)}\psi(r,x)$$
$$+\frac{1}{r^2f(r)}\partial_x^2\psi(r,x)-\frac{q^2B^2x^2}{r^2f(r)}\psi(r,x) = -\frac{q^2\phi^2(r)}{f^2(r)}\psi(r,x)\ . \quad (6.21)$$

6.4.1 Gauge field solution

Now we proceed to solve the gauge field equation which reads upto first order in the BI parameter

$$\left(1+bB^2+\frac{bB^2}{r^4}\right)\phi''(r)+\frac{2}{r}\left(1+bB^2+\frac{2bB^2}{r^4}-b\phi'^2(r)\right)\phi'(r)=\frac{2q^2\psi^2(r,x)}{f(r)}$$
$$\times\left[1+\frac{3b}{2}\left(\frac{B^2}{r^4}-\phi'^2(r)\right)\right]\phi(r)\ . \quad (6.22)$$

Changing variables to $z=\frac{r_+}{r}$, we find the matter field and gauge field equations in z coordinate to be

$$\frac{\partial^2\psi(z,x)}{\partial z^2}+\frac{f'(z)}{f(z)}\frac{\partial\psi(z,x)}{\partial z}-\frac{m^2r_+^2}{z^4f(z)}\psi(z,x)+\frac{q^2r_+^2\phi^2(r)}{z^4f^2(z)}\psi(z,x)$$
$$=-\frac{1}{z^2f(z)}\left[\frac{\partial^2\psi(z,x)}{\partial x^2}-q^2B^2x^2\psi(z,x)\right] \quad (6.23)$$

$$\left(1+bB^2+\frac{bB^2z^4}{r_+^4}\right)\frac{d^2\phi(z)}{dz^2}-\frac{2bB^2z^3}{r_+^4}\frac{d\phi(z)}{dz}+\frac{2bz^3}{r_+^2}\left(\frac{d\phi(z)}{dz}\right)^3$$
$$=\frac{2q^2\psi^2(z,x)}{f(z)}\left[\frac{r_+^2}{z^4}+\frac{3b}{2}\left(\frac{B^2}{r_+^2}-\phi'^2(r)\right)\right]\phi(z)\ . \quad (6.24)$$

At $T=T_c$, the matter field $\psi(z)$ vanishes. Putting $\psi(z)=0$ in eq.(6.24), we obtain

$$\frac{d^2\phi(z)}{dz^2}-\frac{2bB^2\frac{z^3}{r_+^4}}{\left(1+bB^2+\frac{bB^2z^4}{r_+^4}\right)}\frac{d\phi(z)}{dz}+\frac{2b\frac{z^3}{r_+^2}}{\left(1+bB^2+\frac{bB^2z^4}{r_+^4}\right)}\left(\frac{d\phi(z)}{dz}\right)^3=0\ . \quad (6.25)$$

The integrating factor of the above equation is $\frac{1}{\sqrt{1+bB^2\left(1+\frac{z^4}{r_+^4}\right)}}$ which converts the above equation to the following form

$$\frac{d\zeta(z)}{dz}=-\frac{2b}{r_+^2}z^3\zeta^3(z) \quad (6.26)$$

where $\zeta(z) = \frac{\phi'(z)}{\sqrt{1+bB^2\left(1+\frac{z^4}{r_+^4}\right)}}$. To solve this equation, we need to impose the asymptotic behaviour of the gauge field which is

$$\phi(z) = \mu - \frac{\rho}{r_+} z \ . \tag{6.27}$$

Now we integrate eq.(6.26) in the interval between boundary and the event horizon, that is $[0, 1]$

$$\int_0^1 \frac{d\zeta(z)}{\zeta^3(z)} = -\frac{2b}{r_+^2} \int_0^1 z^3 dz \tag{6.28}$$

$$\Rightarrow \frac{1}{\zeta^2(1)} = \frac{b}{r_+^2} + \frac{1}{\zeta^2(0)} \ . \tag{6.29}$$

We also integrate eq.(6.26) in the interval $[1, z]$ and use the above relation to get

$$\frac{1}{\zeta^2(z)} = \frac{b}{r_+^2}(z^4 - 1) + \frac{1}{\zeta^2(1)} \tag{6.30}$$

$$\Rightarrow \frac{1}{\zeta^2(z)} = \frac{bz^4}{r_+^2} + \frac{1}{\zeta^2(0)} \ . \tag{6.31}$$

Using the asymptotic behaviour of $\phi(z)$ (6.27), we finally obtain

$$\phi'(z) = -\sqrt{\frac{1 + bB^2\left(1 + \frac{z^4}{r_+^4}\right)}{1 + b\left(B^2 + \frac{\rho^2 z^4}{r_+^4}\right)}} \frac{\rho}{r_+} \ . \tag{6.32}$$

Note that this expression takes into account the effects of the magnetic field coming from both $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}G^{\mu\nu}$ terms. To be precise, the last term in the numerator and denominator arise from the Born part of the theory and the second term in the numerator and denominator arises from the $\vec{E}.\vec{B}$ term in the BI theory. This relation will be used in the subsequent discussion to calculate the critical magnetic field.

6.4.2 The critical magnetic field

Now we turn our attention at the matter field equation near T_c. Employing the separation of variable technique $\psi(z, x) = X(x)R(z)$ and setting $q = 1$, eq.(6.23) takes the form

$$\frac{R''(z)}{R} + \frac{f'(z)R'(z)}{f(z)R(z)} - \frac{m^2 r_+^2}{z^4 f(z)} + \frac{r_+^2 \phi^2(z)}{z^4 f^2(z)} = \frac{1}{z^2 f(z)}\left[-\frac{X''}{X} + B^2 x^2\right] \ . \tag{6.33}$$

This finally gives on separation, the following equation for $X(x)$

$$\left(-\frac{d^2}{dx^2} + B^2 x^2\right) X = \kappa^2 X \ . \tag{6.34}$$

The above equation for $X(x)$ is identified as the Schrödinger equation in one dimension with a B-dependent frequency which leads us to identify $\kappa^2 = (2n+1)B$ where n is an integer. For $n = 0$, we find that $\kappa^2 = B$ and this helps in finding the critical magnetic field.
The radial part of the matter field takes the form

$$R''(z) + \frac{f'(z)}{f(z)}R'(z) + \left(\frac{r_+^2\phi^2(z)}{z^4 f^2(z)} - \frac{m^2 r_+^2}{z^4 f(z)} - \frac{\kappa^2}{z^2 f(z)}\right)R(z) = 0 \; . \tag{6.35}$$

From the above equation and using the fact that $f(1) = 0$, we find

$$R'(1) = -\left(\frac{m^2}{3} + \frac{\kappa^2}{3r_+^2}\right)R(1) \tag{6.36}$$

$$R''(1) = \left[\frac{m^2}{3} + \frac{m^4}{18} + \frac{\kappa^4}{18r_+^4} + \frac{m^2\kappa^2}{9r_+^2} - \frac{\phi'^2(1)}{18r_+^2}\right]R(1) \; . \tag{6.37}$$

The $\phi'^2(1)$ term present in eq.(6.37) incorporates the effects of the nonlinear BI electrodynamics. We now expand $R(z)$ around $z = 1$ which reads

$$\begin{aligned} R(z) &= R(1) - R'(1)(1-z) + \frac{R''(1)}{2}(1-z)^2 + ... \\ &\approx R(1) - R'(1)(1-z) + \frac{R''(1)}{2}(1-z)^2 \; . \end{aligned} \tag{6.38}$$

Substituting the value of $R''(1)$ and $R'(1)$ in eq.(6.38), we find

$$R(z) = \left[1 + \left(\frac{m^2}{3} + \frac{\kappa^2}{3r_+^2}\right)(1-z) + \left(\frac{m^2}{3} + \frac{m^4}{18} + \frac{\kappa^4}{18r_+^4} + \frac{m^2\kappa^2}{9r_+^2} - \frac{\phi'^2(1)}{18r_+^2}\right)\frac{(1-z)^2}{2!}\right]R(1) \; . \tag{6.39}$$

Setting $m^2 = -2$ and equating eq.(6.39) and eq.(6.16) and their derivatives at $z = z_m$, we obtain

$$\frac{J_- z_m}{r_+} = R(1)\left[1 - \left(\frac{2}{3} - \frac{\kappa^2}{3r_+^2}\right)(1-z_m) + \left(-\frac{4}{9} + \frac{\kappa^4}{18r_+^4} - \frac{2\kappa^2}{9r_+^2} - \frac{\phi'^2(1)}{18r_+^2}\right)\frac{(1-z_m)^2}{2}\right] \tag{6.40}$$

$$\frac{J_-}{r_+} = R(1)\left[\frac{2}{3} - \frac{\kappa^2}{3r_+^2} - \left(-\frac{4}{9} + \frac{\kappa^4}{18r_+^4} - \frac{2\kappa^2}{9r_+^2} - \frac{\phi'^2(1)}{18r_+^2}\right)(1-z_m)\right] \; . \tag{6.41}$$

From the above relations, we finally get

$$\kappa^4 + 4\left(\frac{2+z_m^2}{1-z_m^2}\right)r_+^2\kappa^2 + 4\left(\frac{1+2z_m^2}{1-z_m^2}\right)r_+^4 - \phi'^2(1)r_+^2 = 0 \tag{6.42}$$

which in turn implies, using $\kappa^2 = B$

$$B^2 + 4\left(\frac{2+z_m^2}{1-z_m^2}\right)r_+^2 B + 4\left(\frac{1+2z_m^2}{1-z_m^2}\right)r_+^4 - \phi'^2(1)r_+^2 = 0 \; . \tag{6.43}$$

This equation has exactly the same form as derived in [146]. However, $\phi'^2(1)$ contains the effect of the BI theory and differs from that in [146]. The last term in the above equation upto $\mathcal{O}(b)$ can be obtained from eq.(6.32) and reads

$$\phi'^2(1) = \left[1 + \frac{b}{r_+^4}\left(B^2 - \rho^2\right)\right]\frac{\rho^2}{r_+^2} \ . \tag{6.44}$$

Substituting eq.(6.18) and eq.(6.44) in eq.(6.43), we get upto order $\mathcal{O}(b)$

$$\left(1 - b\frac{\rho^2}{r_+^4}\right)B^2 + 4a_2 r_+^2 B + \tilde{\beta}^2 r_+^4 - \rho^2\left(1 - b\frac{\rho^2}{r_+^4}\right) = 0 \tag{6.45}$$

where $a_2 = \frac{2+z_m^2}{1-z_m^2}$. The solution of the above equation reads

$$B_c = \frac{1}{\left(1 - b\frac{\rho^2}{r_+^4}\right)}\left[\sqrt{4a_2^2 r_+^4 - \left(1 - b\frac{\rho^2}{r_+^4}\right)\left\{\tilde{\beta}^2 r_+^4 - \rho^2\left(1 - b\frac{\rho^2}{r_+^4}\right)\right\}} - 2a_2 r_+^2\right] . \tag{6.46}$$

Now let us denote $T_c \equiv T_c(B)$, then from eq.(6.17) and eq.(6.9) we find

$$\frac{\rho^2}{r_+^4} = \tilde{\beta}^2\{1 + 2b\tilde{\beta}^2(1 - z_m)\}^2\frac{T_c^4(0)}{T^4} \tag{6.47}$$

$$= \tilde{\beta}^2\{1 + 4b\tilde{\beta}^2(1 - z_m) + \mathcal{O}(b^2)\}\frac{T_c^4(0)}{T^4} \ . \tag{6.48}$$

Substituting the above equation and $r_+ = \frac{4\pi}{3}T$ in eq.(6.46), we finally obtain

$$\begin{aligned} B_c &= \frac{16\pi^2\tilde{\beta}T_c^2(0)}{3\left(1 - b\tilde{\beta}^2\frac{T_c^4(0)}{T^4}\right)}\left[\sqrt{1 + \left(\frac{4a_2^2}{\tilde{\beta}^2} - 1\right)\frac{T^4}{T_c^4(0)} + b\tilde{\beta}^2\left(5 - 4z_m - 2\frac{T_c^4(0)}{T^4}\right)} - \frac{2a_2}{\tilde{\beta}}\frac{T^2}{T_c^2(0)}\right] \\ &\approx (1 + b\tilde{\beta}^2)B_0 + \left[\frac{8\pi^2 b\tilde{\beta}^3}{9}\frac{(3 - 4z_m)\,T_c^2(0)}{\sqrt{1 + \left(\frac{4a_2^2}{\tilde{\beta}^2} - 1\right)\frac{T^4}{T_c^4(0)}}}\right] \end{aligned} \tag{6.49}$$

where

$$B_0 = B_c|_{b=0} = \frac{16\pi^2}{9}\tilde{\beta}T_c^2(0)\left[\sqrt{1 + \left(\frac{4a_2^2}{\tilde{\beta}^2} - 1\right)\frac{T^4}{T_c^4(0)}} - \frac{2a_2}{\tilde{\beta}}\frac{T^2}{T_c^2(0)}\right] . \tag{6.50}$$

We observe that the critical magnetic field B_c incorporates the effects of the BI parameter b. Note that the critical magnetic field upto $\mathcal{O}(b)$ differs from that obtained in the Born theory. In the presence of Born electrodynamics, the critical magnetic field reads [146]

$$\begin{aligned} B_c^{(Born)} &= \frac{16\pi^2\tilde{\beta}}{9}fT_c^2(0)\left[\sqrt{\frac{1}{1 + b\tilde{\beta}^2 f^2\frac{T_c^2(0)}{T^2}} + \frac{A_1}{\tilde{\beta}^2}\frac{T^4}{T_c^4(0)}} - \frac{A_2}{\tilde{\beta}}\frac{T^2}{T_c^2(0)}\right] \\ &\approx B_0 + \left[\frac{8\pi^2 b\tilde{\beta}^3}{9}\frac{(3 - 4z_m)\,T_c^2(0)}{\sqrt{1 + \left(\frac{4a_2^2}{\tilde{\beta}^2} - 1\right)\frac{T^4}{T_c^4(0)}}}\right] \end{aligned} \tag{6.51}$$

where

$$f = 1 + 8b\frac{1+2z_m^2}{1+z_m}, \quad A_1 = \frac{12(1+z_m^2+z_m^4)}{f^2(1-z_m^2)^2}, \quad A_2 = \frac{2(2+z_m^2)}{f(1-z_m^2)} . \tag{6.52}$$

Comparing eq.(6.49) and eq.(6.51), we observe that the second term in eq.(6.49) is an extra piece which arises due to the BI theory together with the fact that the gauge field equation has been solved taking into account the effect of the magnetic field. Using eq.(s)(6.17,6.49,6.51) and fixing $z_m = 0.5$, we now compare the critical magnetic field of Born electrodynamics and BI electrodynamics for different values of b at temperature $T = 0$ in Table 6.1. Here we observe that the critical magnetic field

Table 6.1: Comparison of the critical magnetic fields of Born and BI theory at $T = 0$ for $z_m = 0.5$

Value of b	For Born electrodynamics		For BI electrodynamics	
	$\frac{B_c}{T_c^2(0)}$	B_c	$\frac{B_c}{T_c^2(0)}$	B_c
b=0.00	49.6275	1.001ρ	49.6275	1.001ρ
b=0.01	51.6126	0.969ρ	55.5828	1.043ρ
b=0.02	53.5976	0.934ρ	61.5381	1.072ρ

increases in the BI theory compared to that in the Born theory. It is also observed that the critical magnetic field (B_c) for BI electrodynamics increases for higher values of b where as the critical magnetic field for Born theory decreases for higher values of b. The presence of both the Lorentz invariant quantities in BI theory, in particular the extra term $F^{\alpha\beta}G_{\alpha\beta}$ plays a crucial role in the behavior of the critical magnetic field of holographic superconductors. We have also plotted the critical magnetic field vs temperature for different values of BI parameter b in Figure 6.2 which clearly shows that $\frac{B_c}{T_c^2(0)}$ increases for higher values of b. This clearly indicates that the extra $\vec{E}.\vec{B}$ term present in the BI theory is favourable for the Meissner effect as it increases the critical magnetic field at which the superconductivity order gets destroyed.

6.5 Conclusions

In this chapter, we have studied the effects of magnetic field on holographic superconductors by considering Born-Infeld electrodynamics [39]. First, we have discussed about Meissner effect which helps to distinguish type-I and type-II superconductors. Using conventional theory in condensed matter physics, we are able to describe Meissner effect in ordinary superconductor materials. However, it is difficult to apply this conventional method for high T_c superconductors which are mostly unconventional superconductors (cuprate layer based superconductors). For that an alternative tool was established via gauge/gravity duality, which is known as holographic superconductors. It is also found that cuprate layer based superconductors show non-linear Meissner effect [162, 163]. To incorporate the non-linearity in holographic superconductor model, we consider Born-Infeld electrodynamics instead of Maxwell electrodynamics. The investigation is important in its own right as most

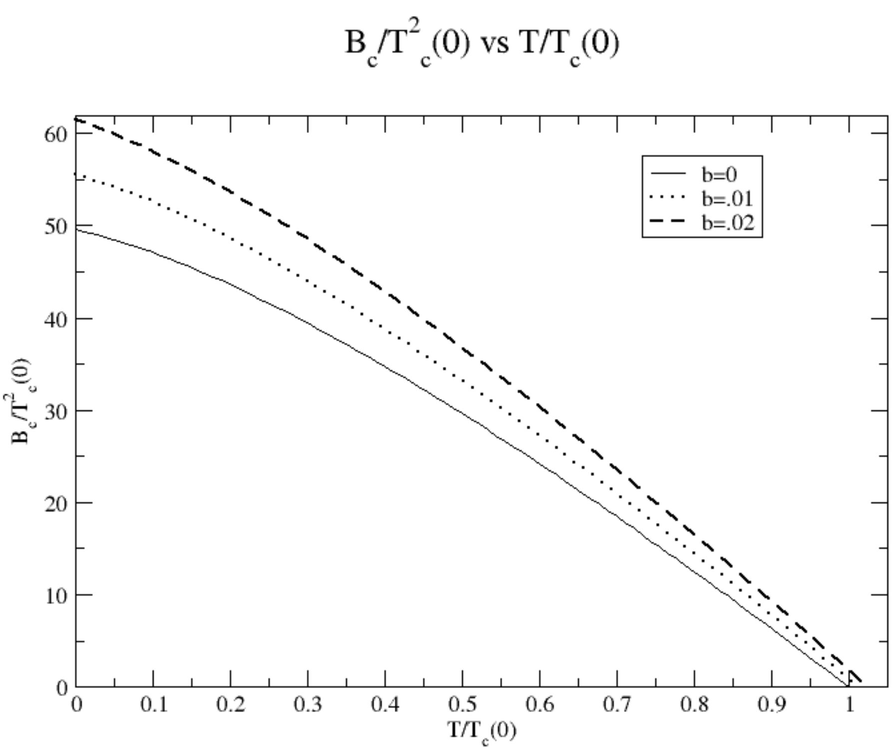

Figure 6.2: $\frac{B_c}{T_c^2(0)}$ vs $\frac{T}{T_c}$ plot for different values of b ($b = 0, b = 0.01, b = 0.02$)

of the studies carried out so far in the literature is with Maxwell electrodynamics. However, the non-linear Messiner like effect has been studied using Born electrodynamics which shows vacuum birefringence. So it is important to consider Born-Infeld electrodynamics for studying non-linear Meissner like effect in holographic superconductors. The study is important in its own right because of distinct advantage of the Born-Infeld electrodynamics over Born electrodynamics, namely, the absence of vacuum birefringence in Born-Infeld theory. Further, it is surely interesting to investigate whether the presence of the extra $\vec{E}.\vec{B}$ term in the Born-Infeld theory favours the Meissner like effect in holographic superconductors over the Born theory. More precisely, the importance of this study lies in the fact that the Born-Infeld theory of non-linear electrodynamics has an extra $\vec{E}.\vec{B}$ which is non-zero only when a magnetic field is switched on. This extra term plays a crucial role in the investigation of the effect of magnetic field on holographic superconductors which is clearly shown in Table 6.1 and Figure 6.2. In the absence of this $\vec{E}.\vec{B}$ term, the critical magnetic field decreases with increase in the value of the Born parameter b thereby indicating that the Born theory does not favour Meissner like effect in holographic superconductors. In contrast, we observe from our analysis that the critical magnetic field increases with increase in the Born-Infeld parameter and its value is greater than that obtained in Born electrodynamics [146] and Maxwell electrodynamics. This indicates that the Born-Infeld theory of non-linear electrodynamics is favourable for Meissner like effect in holographic superconductors.

6.6 Appendix

Here we briefly sketch the matching method technique to obtain the relation between the critical temperature and the charge density. The technique is to match the asymptotic behavior of fields with the horizon behaviour of fields at an arbitrary point z_m between horizon and the AdS boundary. First, we expand the scalar and matter fields near the horizon ($z = 1$)

$$\phi(z) = \phi(1) - \frac{\phi'(1)}{1!}(1-z) + \frac{\phi''(1)}{2!}(1-z)^2 + ... \tag{6.53}$$

$$\psi(z) = \psi(1) - \frac{\psi'(1)}{1!}(1-z) + \frac{\psi''(1)}{2!}(1-z)^2 + ... \, . \tag{6.54}$$

Using fact that $f(1) = 0$, $f'(1) = -3r_+^2$, $f''(1) = 6r_+^2$ together with regularity condition $\phi(1) = 0$, we can find from the gauge field equation

$$\phi''(1) = -\left[\frac{2b}{r_+^2}\phi'^2(1) + \frac{2}{3}\psi^2(1)\left(1 - \frac{b}{r_+^2}\phi'^2(1)\right)^{3/2}\right]\phi'(1) \, . \tag{6.55}$$

Similary from equation for the matter field, we find

$$\psi'(1) = -\frac{m^2}{3}\psi(1), \quad \psi''(1) = \left(\frac{m^4}{18} + \frac{m^2}{3} - \frac{\phi'^2(1)}{18r_+^2}\right)\psi(1) \, . \tag{6.56}$$

Substituting the above expressions in eq.(s)(6.53),(6.54), we get

$$\phi(z) \approx -\left[(1-z) + \left\{\frac{b}{r_+^2}\phi'^2(1) + \frac{1}{3}\psi^2(1)\left(1 - \frac{b}{r_+^2}\phi'^2(1)\right)^{3/2}\right\}(1-z)^2\right]\phi'(1) \tag{6.57}$$

$$\psi(z) \approx \left[1 + \frac{m^2}{3}(1-z) + \frac{1}{2}\left(\frac{m^4}{18} + \frac{m^2}{3} - \frac{\phi'^2(1)}{18r_+^2}\right)(1-z)^2\right]\psi(1) \, . \tag{6.58}$$

Setting $m^2 = -2$ in the above equations, we then match the above behaviour of the scalar and matter fields near horizon with those in the asymptotic region at $z = z_m$. The same thing is carried for their derivatives also. This yields

$$\mu - \frac{\rho}{r_+}z_m = \beta(1-z_m) + \beta\left[\frac{b\beta^2}{r_+^2} + \frac{\alpha^2}{3}\left(1 - \frac{b\beta^2}{r_+^2}\right)^{3/2}\right](1-z_m)^2 \tag{6.59}$$

$$\frac{\rho}{r_+} = \beta + 2\beta\left[\frac{b\beta^2}{r_+^2} + \frac{\alpha^2}{3}\left(1 - \frac{b\beta^2}{r_+^2}\right)^{3/2}\right](1-z_m) \tag{6.60}$$

where $\beta = -\phi'(1)$ and $\alpha = \psi(1)$. Using $T = \frac{3r_+}{4\pi}$ and using the above equations, we get

$$\alpha^2 = \frac{3\left(1 + 2b\tilde{\beta}^2(1-z_m)\right)}{2(1-z_m)(1-b\tilde{\beta}^2)^{3/2}}\left(\frac{T_c^2}{T^2} - 1\right) \tag{6.61}$$

where

$$T_c = \frac{3}{4\pi} \frac{\sqrt{\rho}}{\sqrt{\tilde{\beta}\{1 + 2b\tilde{\beta}^2(1 - z_m)\}}} \tag{6.62}$$

with $\tilde{\beta} = \frac{\beta}{r_+}$. Treating the matter field sector in a similar way yields

$$\frac{J_-}{r_+} z_m = \frac{\alpha}{3} + \frac{2\alpha}{3} z_m - \frac{\alpha}{9}\left(2 + \frac{\beta^2}{4r_+^2}\right)(1 - z_m)^2 \tag{6.63}$$

$$\frac{J_-}{r_+} = \frac{2\alpha}{3} + \frac{\alpha}{9}\left(4 + \frac{\beta^2}{2r_+^2}\right)(1 - z_m) \ . \tag{6.64}$$

The above relations give

$$\tilde{\beta} = \frac{\beta}{r_+} = 2\sqrt{\frac{1 + 2z_m^2}{1 - z_m^2}} \ . \tag{6.65}$$

Chapter 7

Summary and Future directions

The application of gauge/gravity duality in the strongly coupled system has been investigated in this thesis. The gauge/gravity duality has been extracted from string theory which provides a possible framework for unification of four fundamental interactions in Nature. The constituent of matter is made up of strings instead of point-particle and all interactions are supposed to occur by splitting and joining of strings. In string theory, the surface on which open string end, is known as D-brane. D-brane physics allow us to make a map between $\mathcal{N} = 4$ super Yang-Mills (SYM) SU(N) gauge theory and classical supergravity theory in $AdS_5 \times S^5$ [4]. Later, this map is established as AdS/CFT dictionary which has been constructed in more general way [5]-[7]. The conventional tools are not helpful to study strongly coupled gauge theory since perturbation tool does not work in strongly coupled system. The duality provides a tool to analyze a gauge theory (strongly coupled) by investigating gravity theory (weakly coupled).

The phase transition in the different strongly coupled systems has been investigated using phase transition in gravity theory. The Hawking-Page phase transition is the first order phase transition in gravity theory which is associated with the confinement/de-confinement phase transition in QCD [9] (gauge theory). The dual description of the second order phase transition in gravity is associated with the phase transition in strongly coupled superconductors. The asymptotic AdS spacetime allows us to find the second order phase transition in gravity theory since it allows the formation of scalar hair outside of a black hole. Using phase transition from one black hole spacetime to another black hole spacetime, the phase transition in high T_c superconductors (strongly coupled) has been explained. The dual description of these superconductors are known as 'Holographic superconductors' since dual theory lives in one higher spatial dimension. This duality also provides us a tool to study the transport properties of strongly coupled field theory with help of dual description of hydrodynamics. The ratio between the shear viscosity and the entropy density has been found as a universal value $\frac{1}{4\pi}$ in this direction. Entanglement entropy has been computed by minimal area prescription in gravity model.

In this thesis, we have investigated higher dimensional holographic superconductors in the presence of Born-Infeld electrodynamics with three different analytical techniques, namely, Sturm-Liouville eigenvalue method [35], matching method [36] and thermodynamic geometry approach [36, 37]. In Sturm-Liouville eigenvalue method, we recast the equation for condensation in Sturm-Liouville eigenvalue form to know the eigenvalue of this equation which is needed to get the relation between the critical temperature and the charge density. In the matching method, the asymptotic behavior of field is matched with the horizon behavior of field to know the critical temperature. Using the basic set-up of holographic superconductors, the holographic free energy density of the system can be calculated analytically in terms of temperature and the charge density. From holographic free energy density, the thermodynamic metric can be computed using the formalism of the geometrical structure of a thermodynamic system in equilibrium. From this geometrical structure, one can compute the Riemannian scalar curvature using Ruppeiner formalism [156]. The critical phenomena of a system can be analyzed from the divergence of this scalar curvature which provides the critical temperature of holographic superconductors. Born-Infeld electrodynamics is non-linear electrodynamics which introduces non-linearity in the system. The main motivation for considering this electrodynamics is to know the effect of higher derivative gauge correction in the phase transition. We know that higher derivative curvature correction is also affecting the phase transition which motivates us to consider Gauss-Bonnet gravity in the context of phase transition in gravity theory. Let us summarize all findings in our works for this Ph.D thesis.

We have started with a brief description of basic ingredients such as general relativity and Born-Infeld electrodynamics in Chapter 1 and AdS/CFT correspondence in Chapter 2. In Chapter 2, we have first given a brief discussion on AdS spacetime, CFT and string theory which are essentials for better understanding the AdS/CFT correspondence. The AdS/CFT dictionary is then established using GKPW prescription and some application of this dictionary has been discussed. We have also mentioned some basic ingredient in every chapter before the discussion of our work. Our works have been presented from Chapter 3 onward. In Chapter 3, phenomenological theory of superconductivity and the BCS theory of superconductivity has been reviewed for completeness.

Holographic superconductor in presence of BI electrodynamics in background of Gauss-Bonnet gravity in arbitrary spacetime dimension has been analyzed analytically in Chapter 3. In this analysis, we have incorporated the backreaction of the spacetime on the matter field. We have calculated analytically the critical temperature and the condensation operator value of holographic superconductor. These analytical results match with the numerical findings in the literature. The effects of BI parameter, GB parameter, backreaction parameter and spacetime dimension on the condensation has been investigated in this analysis using Sturm-Liouville (SL) eigenvalue method. We have shown that higher value of all these parameters is unfavourable for condensation. Increasing value of these parameters decreases the critical temperature value for fixed charge density of the holographic superconductors. We have shown all results in Tables as well as we have graphically represented our findings.

In Chapter 4, we have investigated the holographic free energy and thermodynamic geometry of holographic superconductor in presence of different electrodynamics, namely, Maxwell electrodynamics (linear electrodynamics) and BI electrodynamics (non-linear electrodynamics). To know the critical temperature and the condensation operator value, we have matched the two behaviors (horizon behavior and asymptotic behavior) of fields at any arbitrary point between horizon and boundary, which is known as 'matching method' technique. From the on-shell action of our holographic superconductors model, we have computed the holographic free energy density of the system. Using the thermodynamic geometry formalism, the thermodynamic metric has been derived from the holographic free energy density. From this geometry structure, we have computed the Riemannian scalar curvature which gives a handle to study critical phenomena. The divergence of the scalar curvature leads us to find the critical temperature of the holographic superconductor which matches with numerical findings in literature. These analytical results also agree with the findings in other analytical approaches. We have shown the effect of non-linear electrodynamics in holographic free energy and thermodynamic geometry.

The critical phenomena of holographic superconductors with BI electrodynamics has been discussed in Chapter 3 and Chapter 4. An analytic computation of conductivity of holographic superconductors in the framework of BI electrodynamics has so far been missing in the literature. In Chapter 5, the conductivity of holographic superconductor in presence of Born-Infeld electrodynamics has been investigated analytically [38]. First, we have obtained the analytical expression for conductivity which has two parts (imaginary and real). The imaginary part of conductivity expression contains a pole at frequency $\omega = 0$ which indicates that real part of conductivity diverges at frequency $\omega = 0$. Therefore, DC conductivity of our holographic superconductor is infinite. This analytical investigation helps us to know the energy band gap of the system. We have calculated the backreacted metric to incorporate BI parameter in metric which leads us to know the effect of BI parameter in estimating energy gap. We have shown that the energy gap increases with increasing value of BI parameter in presence of backreaction. In the Maxwell limit, we have recovered the result which agrees with numerical finding as well as experimental finding.

The BI electrodynamics (known as DBI electrodynamics in literature) and Born electrodynamics (known as BI electrodynamics in literature) is same only when the magnetic part is switched off. The electrodynamics sector of the theory was introduced by Born and Infeld in [58, 59] and it was Dirac [60] who constructed the Hamiltonian formulation of the theory. This misguides the name of these electrodynamics in literature. The difference between these two electrodynamics is because of the extra $\vec{E}.\vec{B}$ term in the BI theory which is absent in the Born theory. When magnetic part is switch on, the difference is apparent. In Chapter 6, we have analyzed Meissner like effect of our holographic superconductor model. The study of non-linear effects on the critical magnetic field in the presence BI electrodynamics has not been carried out so far in the literature whereas the response of holographic superconductor to the external magnetic field has been investigated in presence of Born electrodynamics in [146]. Using matching method, we have obtained the analytical expression of the critical magnetic field in presence of BI electrodynamics. We have shown the value of the critical magnetic field in a Table in terms of the critical temperature and plotted the graph the critical magnetic field as function the temperature. The critical magnetic field increases with increasing value of BI parameter whereas this value decreases with increasing value of the Born parameter.

In summary, we have investigated all basic properties of holographic superconductor in presence of BI electrodynamics which is s-wave holographic superconductor. This investigation gives some overall views of strongly coupled superconductors from dual gravity model which are nothing but a toy model of strongly coupled superconductors. This mathematical tool was established to analyze the superconductivity in CuO plane which are high T_c superconductors. To analyze such high T_c superconductors, this mathematical tool is very helpful although this analysis does not give any details of the mechanism of superconductivity in real world materials. To understand such materials we need to construct a gravity dual which are more realistic model for high T_c superconductors. These superconductors exhibit d-symmetry. In future direction, we are planning to study analytically d-wave holographic superconductors which are more realistic high T_c superconductors. More study on dual gravity theory of superconductors will lead us to know better characteristic of such real world materials. We will also try to investigate other applications of gauge/gravity duality such as holographic QCD, holographic entanglement entropy and holographic complexity. Applications of this duality may lead us one day to find the quantum theory of gravity.

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